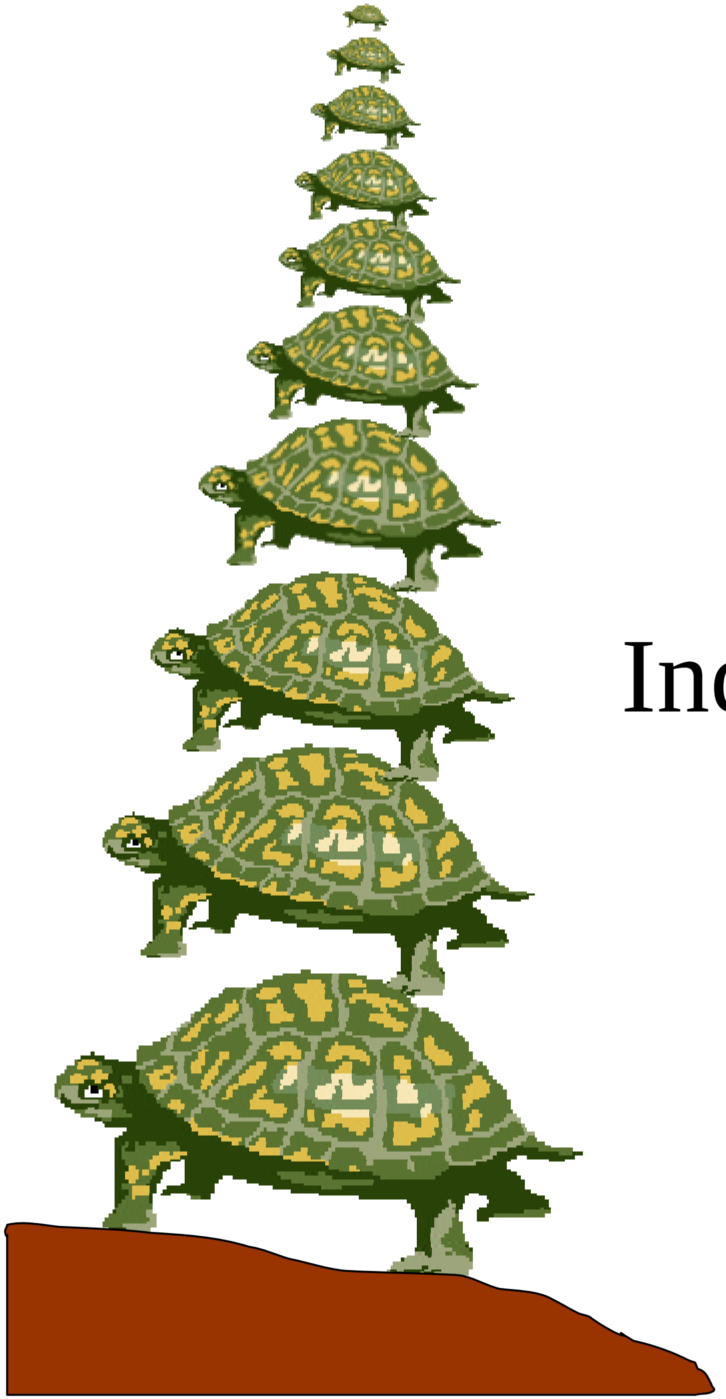


Induction & Recursion

Weiss: ch 7.1

- Recursion
  - a programming strategy for solving large problems
  - Think “divide and conquer”
  - Solve large problem by splitting into smaller problems of same kind
- Induction
  - A mathematical strategy for proving statements about large sets of things
- First we learn induction…

Functions

- Example: Let *S:int* ? *int* be a function such that S(n) is the sum of natural numbers from 0 to n.
  - Iterative form: S(n) = 0+1+…+n
  - Closed form: S(n) = n(n+1)/2
- Can we prove equality?
  - Theorem: For any value of n in N, S(n) = n(n+1)/2

Proving Theorem for all N : Clever Tricks

A Second Example: Sum of Squares

- Example: Let *Q:int* ? *int* be (iterative form)

  $Q(n) = 1^2 + 2^2 + 3^2 + \ldots + n^2$

- Closed Form?

  Intuition: sum of integers → quadratic $(n^2 + n)/2$

  Guess: sum of squares → cubic $(an^3 + bn^2 + cn + d)$

- Coefficients:

  a = 1/3, b = ½, c = 1/6, d = 0

  $Q_{CF}(n) = n^3/3 + n^2/2 + n/6 = n(n+1)(2n+1)/6$

- Conjecture:

  $P(n)$: $Q(n) = Q_{CF}(n)$

- But how to prove P(n) for all n?

Recursive Form

- Rewrite Q as:

  $Q(n) = 1^2 + 2^2 + 3^2 + \ldots + (n-1)^2 + n^2$

  $= Q(n-1) + n^2$

- Each term defined as a function of the one before it

  recursive case: $Q_{RF}(n) = Q_{RF}(n-1) + n^2$

- We still need a 'first' term

  base case: $Q_{RF}(1) = 1^2 = 1$

Evaluating a Recursive Definition

- $Q(5) = Q(4) + 5^2 = 30 + 25 = 55$

  $Q(4) = Q(3) + 4^2 = 14 + 16 = 30$

  $Q(3) = Q(2) + 3^2 = 5 + 9 = 14$

  $Q(2) = Q(1) + 2^2 = 1 + 4 = 5$

  $Q(1) = 1$ *(base case)*

Induction: A Bizzare Example[1]

- Consider a planet X, where the following rule holds:

  "If it rains one day, it also rains the next day"

- Consider two scenarios.

[1]Adapted from http://www-math.utsc.utoronto.ca/calculus/Redbook/goldch1.pdf

Scenario A

- You land on planet X and
  it does **not** rain on the day you arrive.

- What can you conclude?

a) It will never rain on planet X.

b) It has never rained on planet X.

c) It did not rain yesterday on planet X.

d) It will rain tomorrow on planet X.

Scenario B

- You land on planet X and
  it **does** rain on the day you arrive.

- What can you conclude?

e) It rains every day on planet X.

f) It will rain tomorrow on planet X.

g) It rained yesterday on planet X.

h) It will rain every day from now on.

Induction

- Mathematical argument consisting of:
  - A base case: A particular statement, say P(1), that is true.
  - An inductive hypothesis: Assume we know P(n) is true.
  - An inductive step: If we know P(n) is true, we can infer that P(n+1) is true.

Proof of C(n): $Q(n) = Q_{CF}(n)$

- Base case:
  $Q(1) = 1 = 1(1+1)(2*1+1)/6 = Q_{CF}(1)$
  so P(1) holds.
- Inductive hypothesis:
  Let us assume P(n). That is,
  $Q(n) = Q_{CF}(n) = (n+1)(2n+1)/6$
- Inductive step:
  $Q(n+1) = Q(n) + (n+1)^2$ *from recursive form*
  $= n(n+1)(2n+1)/6 + (n+1)^2$ *from i.h.*
  $= (n+1)(n+2)(2n+3)/6$ *from algebra*
  $= Q_{CF}(n)$
  Thus P(n+1) holds.
- Conclusion: $Q(n) = Q_{CF}(n)$ for all n >= 1

Say What?

Proof by induction that P(n) for all n:

- P(1) holds, because ….
- Let's assume P(n) holds.
- P(n+1) holds, because …
- Thus, *by induction*, P(n) holds for all n.

- Your job:
  - Choose a good property P(n) to prove.
    - hint: deciding what *n* is may be tricky
  - Copy down the proof template above.
  - Fill in the two '…'
  - Relax.

Sum of Integers (redux)

- Conjecture

  P(n): The sum S(n) of the first n integers is equal to n(n+1)/2.
- Recursive Form:
- Proof by induction:

More Examples

- Prove for all $n \geq 1$, that 133 divides $11^{n+1}+12^{2n-1}$.
- P(n) =
- No recursive form here…
- Proof by induction…

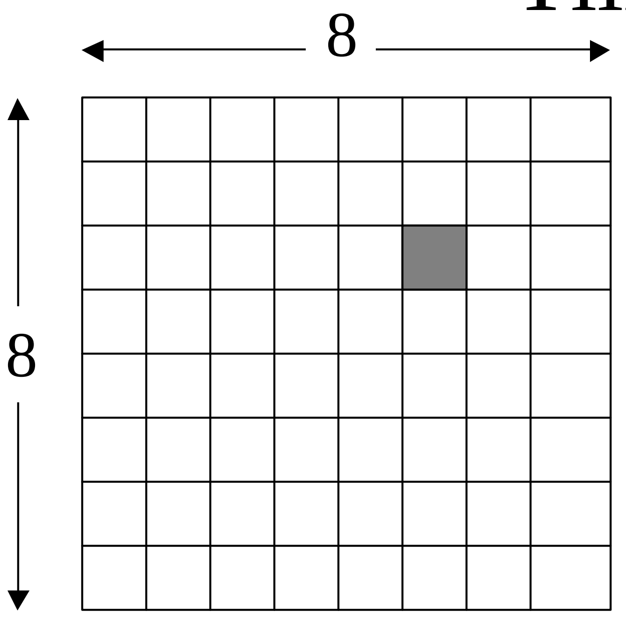
More Examples

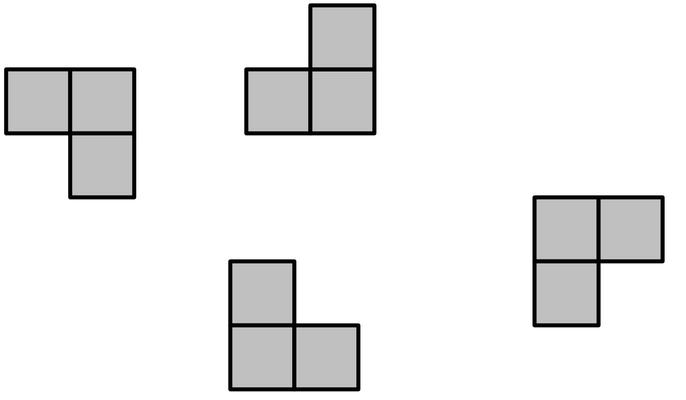
- Prove that $n! > 2^n$ for all integers $n \geq 4$.
- $P(n) =$
- No recursive form here…
- Proof by induction…

Induction (Variation 1)

- We don’t have to start with P(1) only.
  - Often start with P(0), and prove for all n >= 0
  - Can start with P(n_0), and prove for all n >= n_0 (does not say anything about 0, 1, …, n_0-1 though!)
  - Can start with several base cases, P(0), P(1), .., P(k), then use induction for n >= k.

Tiling Problem





- Problem:
  - –Chess board with one square missing
  - –Can we fill remainder with “L” shaped tiles?

Sanity check: 64 squares remaining, divisible by 3

Use Induction

- What is *n*?
  - Number of squares missing?
  - Number of tiles already placed?
  - Number of squares remaining?
- Consider a 2^n by 2^n board

Base case

4x4 case

General Case

Induction Gone Awry

- **Definition**: If a != b are two positive integers, define $max(a, b)$ as the larger of a or b. If $a = b$ define $max(a, b) = a = b$.
- **Conjecture** $A(n)$: if a and b are two positive integers such that $max(a, b) = n$, then $a = b$.
- **Proof (by induction)**:

  Base Case: $A(1)$ is true, since if $max(a, b) = 1$, then both a and b are at most 1. Only $a = b = 1$ satisfies this condition.

  Inductive Case: Assume $A(n)$ for $n >= 1$, and show that $A(n+1)$. If $max(a, b) = n+1$, then $max(a-1, b-1) = n$. By the inductive hypothesis, $a-1 = b-1$, so $a = b$.
- **Corrollary**: $3 = 5$
- **Proof**: $max(3, 5) = 5$. Since $A(5)$ is true, then $3 = 5$.

Caveats

- When proving something by induction…
  - Often <u>easier</u> to prove a *more general (harder)* problem
  - Extra conditions makes things <u>easier</u> in inductive case
    - You have to prove more things in base case & inductive case
    - But you get to use the results in your inductive hypothesis
    - e.g., tiling for n x n boards is impossible, but 2^n x 2^n works
  - You must verify conditions before using I. H.
- Induction often fails
  - Doesn't mean the property is false
  - Choosing what to prove is usually the hardest part
- Exercises (optional) on web site & in Weiss

Strong Induction (Variation 2)

- Up till now, we used *weak induction*

Proof by (strong) induction that P(n) for all n:

- P(1) holds, because ….
- Let's assume P(m) holds for 1 <= m <= n.
  (that is, P(1) holds, P(2) holds, …, and P(n) holds).
- P(n+1) holds, because …
- Thus, *by induction*, P(n) holds for all n.

Postage Stamps

- Prove that every amount of postage of 12¢ or more can be formed using 4¢ and 5¢ stamps.
  - What is P(n)?
  - What base case(s)?