Mathematical Induction

Readings on induction.

(a) Weiss, Sec. 7.2, page 233

(b) Course slides for lecture and notes recitation.

Every criticism from a good man is of value to me. What you hint at generally is very, very true: that my work will be grievously hypothetical, and large parts by no means worthy of being called induction, my commonest error being probably induction from too few facts. **Charles R. Darwin**

induction vs deduction

Induction: 2a (1) **:** inference of a generalized conclusion from particular instances.

2a (2): a conclusion arrived at by induction

2b: mathematical demonstration of the validity of a law concerning all the ints 0, 1, 2, … by proving that (1) it holds for 0 and (2) if it holds for arbitrary int *k* then it holds for *k*+1 -- called *mathematical induction.*

Deduction: 2a : the deriving of a conclusion by reasoning; an inference in which the conclusion about particulars follows necessarily from general or universal premises --a proof?

2b: a conclusion reached by logical deduction.

Overview

- **Recursion**
  - a strategy for writing programs that compute in a "divide-and-conquer" fashion
  - solve a large problem by breaking it up into smaller problems of same kind
- **Induction**
  - a mathematical strategy for proving statements about integers (more generally, about sets that can be ordered in some fairly general ways)
- **Induction** and **recursion** are intimately related.

Defining Functions

- It is often useful to write a given function in different ways. Example: Let S:int → int be a function, where S(n) is the sum of the natural numbers from 0 to n.

  S(0) = 0, S(3) = 0+1+2+3 = 6

  - One definition: iterative form:
    - S(n) = 0 + 1 + … + n
  - Second definition: recursive form:
    - S(0) = 0
    - S(n) = S(n–1) + n for n ≥ 0
  - Third definition: closed-form:
    - S(n) = n(n+1)/2

Notation for recursive functions

Base case

Sn(0) = 0

S(n) = S(n–1) + n for n > 0

Recursive case

or

S(n+1) = S(n) + (n+1) for n>= 0

Can we show that these two definitions of S(n) are equal?

$S_r(0) = 0$

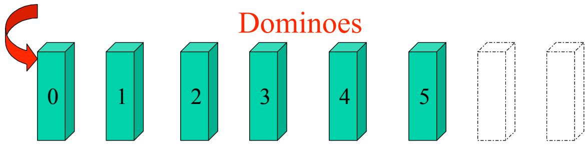
$S_r(n) = S_r(n-1) + n$ for $n > 0$

r: recursive

$S_c(n) = n(n+1)/2$

c: closed-form

Dominoes



- Assume equally spaced dominoes, where spacing between dominoes is less than domino length. Argue that dominoes fall.
- Dumb argument:
  - Domino 0 falls because we push it over.
  - Domino 1 falls: domino 0 falls; it is longer than inter-domino spacing, so it knocks over domino 1.
  - Domino 2 falls: domino 1 falls; it is longer than inter-domino spacing, so it knocks over domino 2.
  - ……
- How do we do this argument nicely?

Mathematical induction

Proof of P(n) for all $n \geq 0$,   where   P(n): Domino n falls.

- **Base case**: Domino 0 falls because we push it over.
- **Inductive case**: Assume **inductive hypothesis** P(k) for any $k \geq 0$ and prove P(k+1):
  - Assume P(k): Domino k falls.
  - Since length of domino k is greater than the inter-domino spacing, it knocks over domino k+1, so domino k+1 false, so P(k+1) holds.

- This is an **inductive** argument.
- This is called **weak** induction. There is also **strong** induction (see later).
- Compact argument, and it works even for an infinite number of dominoes!

Weak induction over integers

Theorem: P(n) holds for all integers $n \geq 0$.

•Proof by (weak) mathematical induction:

Base case: Show that P(0) is true.

Inductive case: Assume **inductive hypothesis** P(k) for any $k \geq 0$ and, using P(k), prove P(k+1).

To prove something by math induction, you HAVE to put in the form:

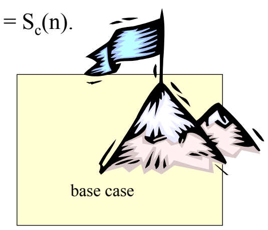
Prove P(n) for all $n \geq 0$.

$S_r(0) = 0$

$S_r(n) = S_r(n-1) + n$

$S_c(n) = n(n+1)/2$

Let P(n) be the proposition that $S_r(n) = S_c(n)$.



Prove P(n), for $n \geq 0$.

Base case: Prove P(0)

$S_c(0)$

= <def of S_c>

0(0+1)/2

= <arithmetic>

0

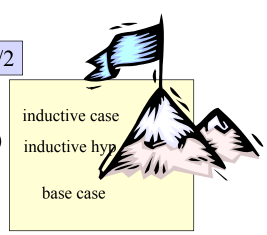
<Def of $S_r(0)$ >

$S_r(0)$

$S_r(0) = 0$

$S_r(n) = S_r(n-1) + n$

$S_c(n) = n(n+1)/2$



Inductive case: Assume P(k), prove P(k+1)

$S_r(k+1)$

= **<def of S_r >**

$S_r(k) + (k+1)$

= **<Use P(k)>**

$S_c(k) + (k+1)$

= **<def of S_c >**

k(k+1)/2 + (k+1)

= **<Arithmetic >**

(k+1)(k+2)/2

= **<Def of S_c >**

$S_c(k+1)$

P(k): $S_r(k) = S_c(k)$

P(k+1): $S_r(k+1) = S_c(k+1)$

$S_r(k+1) = S_r(k+1-1) + k+1$

$S_c(k+1) = (k+1)(k+1+1)/2$

Another example of weak induction

Prove: for all n, $n \geq 0$, P(n) holds, where

P(n): 0 + 1 + 2 + … + n = n(n+1)/2

Base case: Prove P(0):	**Inductive case**: Assume P(k), prove P(k+1)
0(0+1)/2 = <arithmetic> 0 = <0+1+2+…0 = 0> 0 + 1 + 2 + … 0	0 + 1 + 2 + … + (k+1) = <arithmetic> 0 + 1 + 2 + … + k + (k+1) = <Use P(k)> k(k+1)/2 + (k+1) = <Arith> (k+1)(k+1+1)/2

P(k+1): 0 + 1 + 2 + … + (k+1) = (k+1)/(k+1+1)/2

Goal in developing the inductive case: EXPOSE P(k)

Inductive case: Assume P(k), prove P(k+1)

$S_r(k+1)$
= <def of S_r >
$S_r(k) + (k+1)$
= **<Use P(k)>**
$S_c(k) + (k+1)$
= <def of S_c >
$k(k+1)/2 + (k+1)$
= <Arith >
$(k+1)(k+2)/2$
= <Def of S_c >
$S_c(k+1)$

P(k): $S_r(k) = S_c(k)$

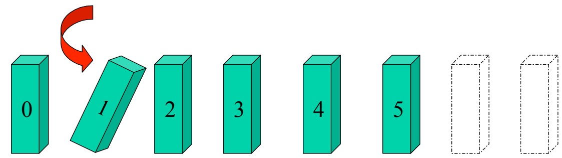
$S_c(n) = n(n+1)/2$

$S_r(0) = 0$

$S_r(n) = S_r(n-1) + n$

Formula above has the LHS of P(k) in it. We have exposed P(k); we have made it possible to use it.

Note on base case



- In some problems, we are interested in showing some proposition is true for integers greater than or equal to some lower bound (say b)
- Intuition: we knock over domino b; dominoes before it are not knocked over. Not interested in dominoes 0,1,…,(b-1).
- In general, base case in induction does not have to be 0.
- If base case is some integer *b*, induction proves proposition for $n = b, b+1, b+2, \ldots$.
- Does not say anything about $n = 0, 1, \ldots, b-1$

Weak induction: non-zero base case

Theorem. for all $n \geq b$, P(n) holds.

Proof.

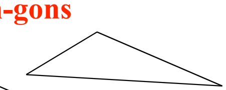
- **Base case**: Prove P(b)
- **Inductive case**: Assume inductive hypothesis P(k) for any $k \geq b$ and prove P(k+1).

or

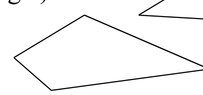
- **inductive case**: Assume inductive hypothesis P(k-1) for $k > b$ and prove P(k).

Proof about n-gons

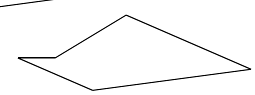
convex polygon (triangle)



convex polygon



not-convex polygon



polygon is **convex** if the line between any two points in it is entirely within the polygon. Or, if all its angles are < 180 degrees.

n-gon: convex polygon with n sides

Theorem. The angles in an n-gon add up to 180(n-2)

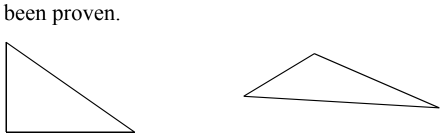
Theorem. For all $n \geq 3$, P(n) holds:

P(n): (sum of angles of an n-gon) = 180(n-2)

Proof.

Base case: P(3): (sum of angles of a 3-gon) = 180(3-2)

We accept as a fact that the sum of the angles of a 3-gon —a triangle— is 180 degrees. Assume it has been proven.



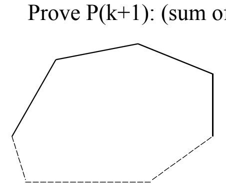
Theorem. The angles in an n-gon add up to 180(n-2)

Theorem. For all $n \geq 3$, P(n) holds:

P(n): (sum of angles of an n-gon) = 180(n-2)

Inductive case: Assume P(k): (sum of angles k-gon) = 180(k-2).

Prove P(k+1): (sum of angles (k+1)-gon) = 180(k-1)



(n+1)-gon

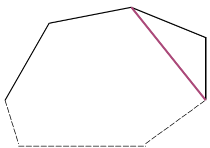
To expose P(k) we have to find an n-gon within the (n+1)-gon. How do we do it?

Theorem. The angles in an n-gon add up to 180(n-2)

Theorem. For all n ≥ 3, P(n) holds:

P(n): (sum of angles of an n-gon) = 180(n-2)

Inductive case: Assume P(k): (sum of angles k-gon) = 180(k-2).
Prove P(k+1): (sum of angles (k+1)-gon) = 180(k-1)



n-gon + 3-gon

sum of angles in (n+1)-gon
= <how the -gons are drawn>
(angle-sum: 3-gon) + (angle-sum: n-gon)
= <P(3) and P(n)>
180 + 180(n-2)
= <Arithmetic>
180(n+1-2)

Using stamps for money

We have lots of 3-cent and 5-cent stamps. Show that any amount of money that is at least 8 cents can be made using 3-cent and 5-cent stamps.

Theorem. For all n ≥ 8, P(n) holds, where

P(n): n can be made using 3-cent and 5-cent stamps.

or

P(n): n = b*3 + c*5 for some natural numbers b and c

Using stamps for money

Theorem. For all n ≥ 8, P(n) holds, where

P(n): n can be made using 3-cent and 5-cent stamps.

Proof.

Base cases: P(8) and P(9).

Make 8 using a 3-cent and a 5-cent stamp.

Make 9 using three 3-cent stamps.

Sometimes, it helps to have more than one base case!!!

Using stamps for money

Theorem. For all n ≥ 8, P(n) holds, where

P(n): n can be made using 3-cent and 5-cent stamps.

Inductive case: Assume inductive hypothesis P(n) for n≥9.
Prove P(n+1):

Case 1: The pile that adds up to n has a 5-cent stamp.
Take out one 5-cent stamp and put in two 3-cent stamps.
Pile now adds up to n+1

Case 2: The pile that adds up to n has no 5-cent stamps.
Since n≥9, the pile has at least three 3-cent stamps. Take three of them out and put in two 5-cent stamps.
Pile no add up to n+1.

Q.E.D. Quit.End.Done.

When induction fails

- Sometimes, an inductive proof strategy for some proposition may fail.
- This does not necessarily mean that the proposition is wrong.
  - It just means that the inductive strategy you are trying fails.
- A different induction or a different proof strategy altogether may succeed.

Strong induction

Prove P(n) for all n≥0.

- **Proof by weak induction**:
  - **Base case**: prove P(0).
  - **Inductive case**: Assume P(k) for k≥0 and prove P(k+1).
- **Proof by strong induction**:
  - **Base case**: prove P(0),
  - **Inductive case**: Assume P(1), P(2), …, P(k) for k≥0 and prove P(k+1).
- **Both techniques are equally powerful (but proof by strong induction is sometimes easier)!!!!**
- **No big deal, whether you use one or the other. So forget about the difference between them!**

Theorem: every int > 1 is divisible by a prime

Definition: n is a prime if $n \geq 2$ and the only positive ints that divide n are 1 and n.

Theorem: For all $n \geq 2$, P(n) holds, where

P(n): n is divisible by a prime.

Proof:

Base cases: P(p) where p is a prime!!!!!!!!
Since p is a prime, it is divisible by itself.

Inductive cases: Prove P(k) for non-prime k, using the inductive hypotheses P(2), P(3), …, P(k-1). Since k is not a prime, by definition, it is divisible by some int b (say) in 2..k-1. P(b) holds, so some prime divides b. Since b divides k, that prime divides k as well. Q.E.D.

Editorial comments

- Induction is a powerful technique for proving propositions.
- We used recursive definition of functions as a step towards formulating inductive proofs.
- However, recursion is useful in its own right.
- There are closed-form expressions for sum of cubes of natural numbers, sum of fourth powers etc. (see any book on number theory).