CS/ENGRI 172, Fall 2003: Computation, Information, and Intelligence
10/3/03: Turing Machine Computability

Turing Machine Universality

Data/Program duality for Turing machines: Turing machine control tables can be written on a Turing machine's tape as input.

The enumerability of Turing machines: We can impose an ordering $M_1, M_2, M_3, \ldots$ on Turing machines.

Universality of Turing machines: There is a universal Turing machine U that takes any program M_i and any input x as input and simulates what M_i would do given input x. That is,

$$U(M_i, x) \begin{cases} \text{outputs } M_i(x) & \text{if } M_i \text{ halts on } x \\ \text{runs forever} & \text{if } M_i \text{ runs forever on } x \end{cases}$$

Simplifying Assumption: Though our results hold in the general case, for ease of argument we will assume that we are only looking at Turing machines whose input is a finite sequence of A's. We will also use the simplifying notation that running M_i on input j A's can be written as $M_i(j)$.

Definition: The *halting function* $h(M_i, j)$ takes as input a Turing machine M_i and an input string of j A's and has value 1 if M_i halts when running on j A's as input, and has value 0 if M_i runs forever when running on j A's an input. That is,

$$h(M_i, j) = \begin{cases} 1 & \text{if } M_i(j) \text{ halts} \\ 0 & \text{if } M_i(j) \text{ runs forever} \end{cases}$$

Uncomputability Theorem: The halting function is not computable. That is, there is no Turing machine that can compute h.

We will prove this by assuming such a Turing machine exists and reaching a contradiction.

Proof Outline:
Definition: The *Universal Termination Detector*, D, is a Turing machine that computes the halting function[1]:

$$D(M_i, j) \text{ outputs } \begin{cases} 1 & \text{if } h(M_i, j) = 1 \\ 0 & \text{if } h(M_i, j) = 0 \end{cases}$$

If D exists, we can build an "evil" Turing machine X using D which takes j A's as input, simulates D on M_j and j as input, and has the following output behavior:

$$X(j) \begin{cases} \text{outputs } 1 & \text{if } D(M_j, j) = 0 \\ \text{runs forever} & \text{if } D(M_j, j) = 1 \end{cases}$$

This evil Turing machine X will allow us to reach a contradiction.

It is D that allows us to build X, so there cannot be a Universal Termination Detector[2]. That is, the halting function is uncomputable.

[1]Remember, D is a Turing machine; it *computes* the halting function, but is not the halting function itself.

[2]Note that this is not a proof that the halting function does not exist. The halting function is a well-defined mathematical object. It is a proof that the halting function cannot be computed by any Turing machine.