Lecture 23

Loop Invariants

Announcements for This Lecture

Prelim 2

- Thursday at 7:30 pm
  - **A–F** in Uris G01
  - **G-H** in Malott 228
  - **I–L** in Ives 305
  - **M-Z** in Statler Aud.
- All review material online
  - Similar to previous years
  - Just changed "hard parts"

Assignments

- A6 due **TOMORROW**
  - Complete it by midnight
  - Also, fill out survey
- A7 due **December 10**
  - Focus of Thursdays lecture
  - 2.5 weeks including T-Day
  - 2 weeks without the break
  - Extensions are possible!
- Both are very important
  - Each worth 8% of grade

Goal For Today

- This lecture is a **programming technique**
  - Completely independent of Python
  - Will learn it again (exactly) in CS 2110
- Useful tool for ensuring **code correctness**
  - Some loops are too complicated to debug
  - Relying on watches/traces not enough
  - This technique helps reduce errors at the start
- Preview of what higher level CS is like

Terminology: Range Notation

- m..n is a range containing n+1-m values
    - 2..5 contains 2, 3, 4, 5. Contains 5+1 – 2 = 4 values
    - 2..4 contains 2, 3, 4. Contains 4+1 – 2 = 3 values
    - 2..3 contains 2, 3. Contains 3+1 – 2 = 2 values
    - 2..2 contains 2. Contains 2+1 – 2 = 1 values
    - 2..1 contains ???

What does 2..1 contain?

A: nothing
B: 2,1
C: 1
D: 2
E: something else

Terminology: Range Notation

- m..n is a range containing n+1-m values
  - 2..5 contains 2, 3, 4, 5. Co lues
  - 2..4 contains 2, 3, 4. Co ues
  - 2..3 contains 2, 3. Co ues
  - 2..2 contains 2. Contains 2+1 – 2 = 1 values
  - 2..1 contains ???

Not the same as range(m,n)

- The notation m..n, always implies that m <= n+1
  - So you can assume that even if we do not say it
  - If m = n+1, the range has 0 values

Assertions: Tracking Code State

- **assertion**: true-false statement placed in a program to *assert* that it is true at that point
  - Can either be a **`comment`**, or an **`assert`** command
- **invariant**: assertion supposed to "always" be true
  - If temporarily invalidated, must make it true again
  - **Example**: class invariants and class methods
- **loop invariant**: assertion supposed to be true before and after each iteration of the loop
- **iteration of a loop**: one execution of its body

Assertions versus Asserts

- Assertions **prevent bugs**
    - Help you keep track of what you are doing
- Also **track down bugs**
    - Make it easier to check belief/code mismatches
- The assert statement is a (type of) assertion
    - One you are **enforcing**
    - Cannot always convert a comment to an assert

 # x is the sum of 1..n

Comment form of the assertion.

The root of all bugs!

x ? n 1

x ? n 3

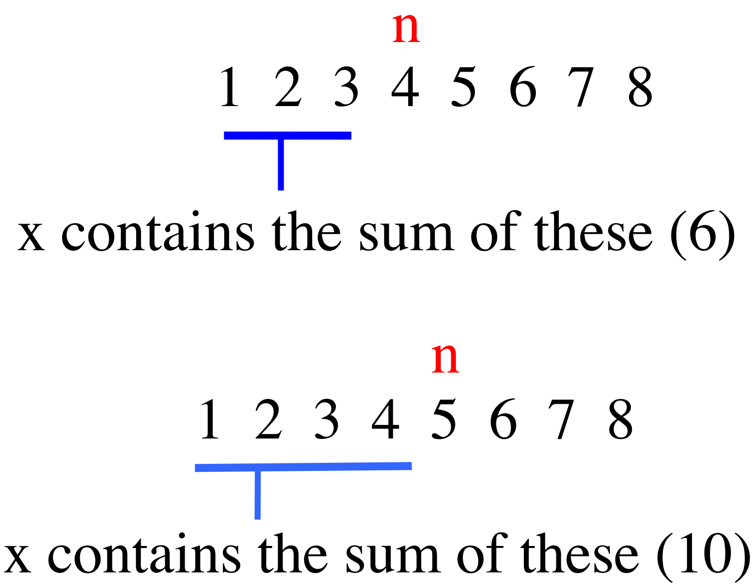
x ? n 0

Preconditions & Postconditions

precondition

```
# x  = sum of 1..n-1
x = x + n
n = n + 1
# x =  sum of 1..n-1
```

postcondition



x contains the sum of these (6)

x contains the sum of these (10)

- **Precondition:** assertion placed before a segment
- **Postcondition:** assertion placed after a segment

Relationship Between Two

If precondition is true, then postcondition will be true

Solving a Problem

precondition

```
# x  = sum of 1..n

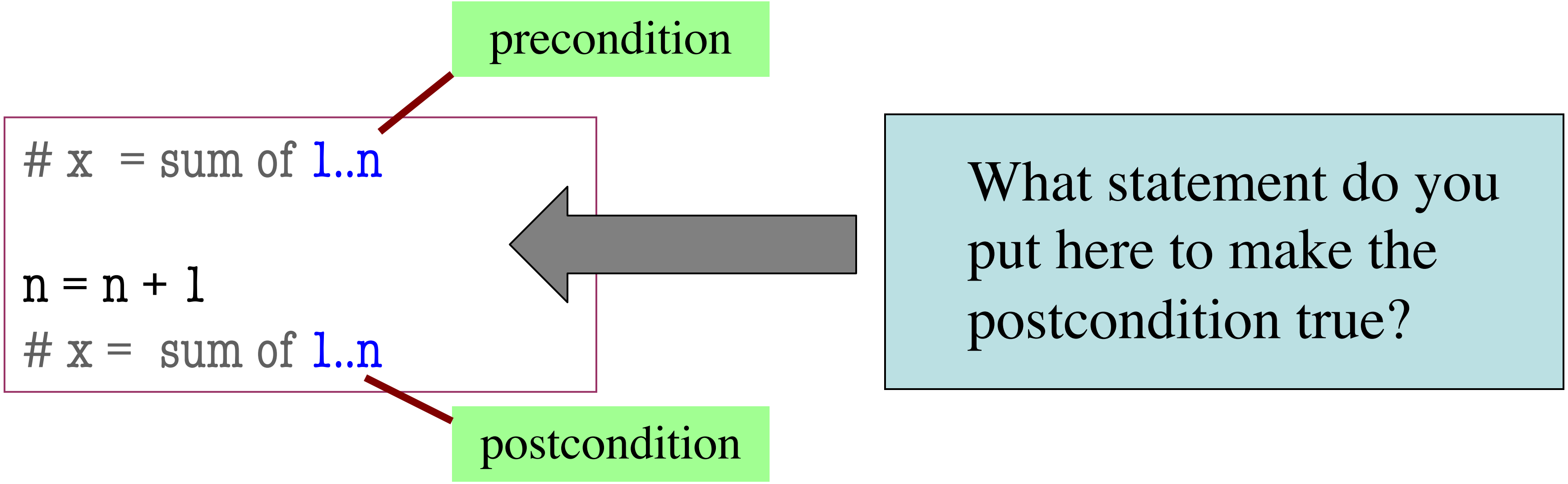
n = n + 1
# x =  sum of 1..n
```

postcondition

What statement do you put here to make the postcondition true?

A: x = x + 1
B: x = x + n
C: x = x + n+1
D: None of the above
E: I don’t know

Solving a Problem



precondition

```
# x  = sum of 1..n

n = n + 1
# x =  sum of 1..n
```

postcondition

What statement do you put here to make the postcondition true?

A: $x = x + 1$

B: $x = x + n$

C: $x = x + n+1$

D: None of the above

E: I don’t know

Remember the new value of n

Invariants: Assertions That Do Not Change

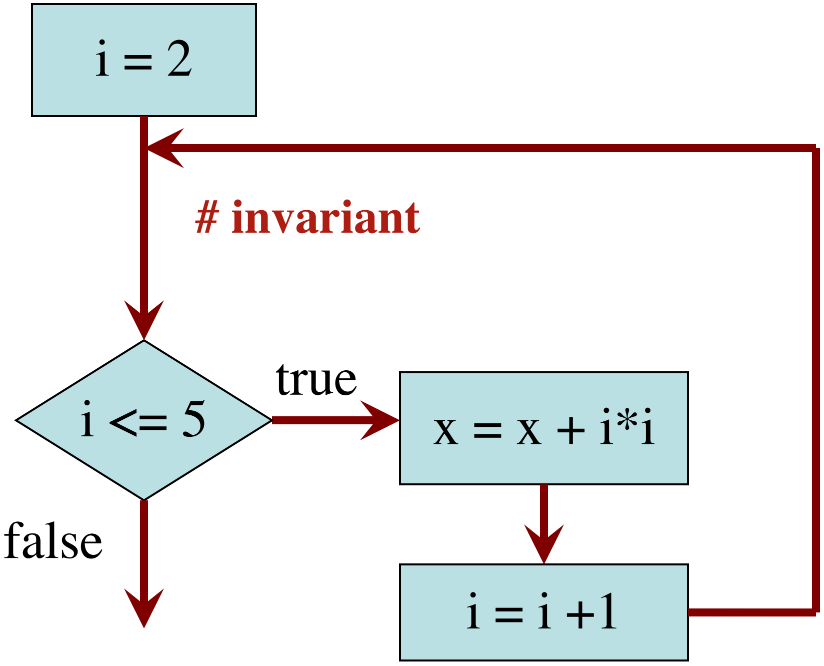
- **Loop Invariant:** an assertion that is true before and after each iteration (execution of repetend)

```
x = 0; i = 2
while i <= 5:
    x = x + i*i
    i = i +1
# x = sum of squares of 2..5
```

Invariant:

x = sum of squares of 2..i-1

in terms of the range of integers that have been processed so far



The loop processes the range 2..5

Invariants: Assertions That Do Not Change

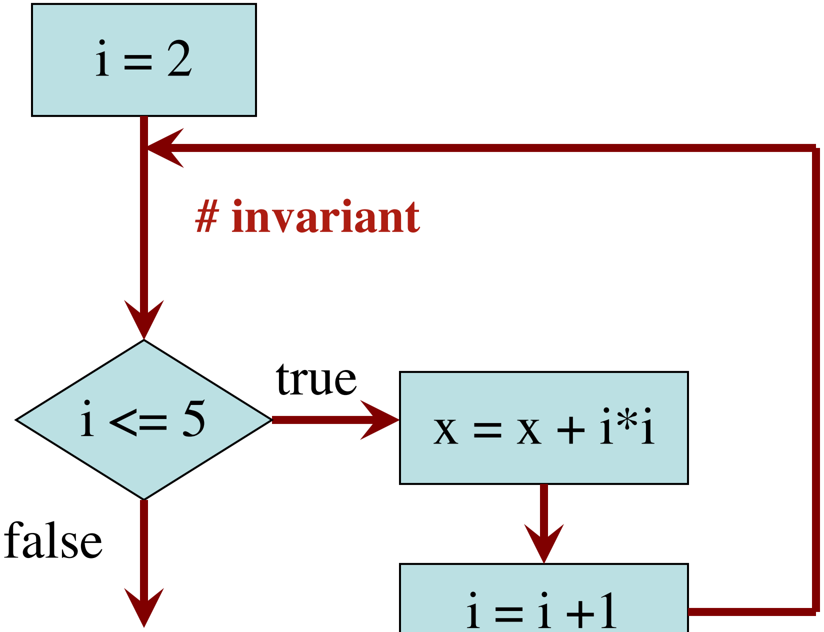
```
x = 0; i = 2
# Inv: x = sum of squares of 2..i-1
while i <= 5:
    x = x + i*i
    i = i +1
# Post: x = sum of squares of 2..5
```

Integers that have been processed:

Range 2..i-1:

x 0

i ?



The loop processes the range 2..5

Invariants: Assertions That Do Not Change

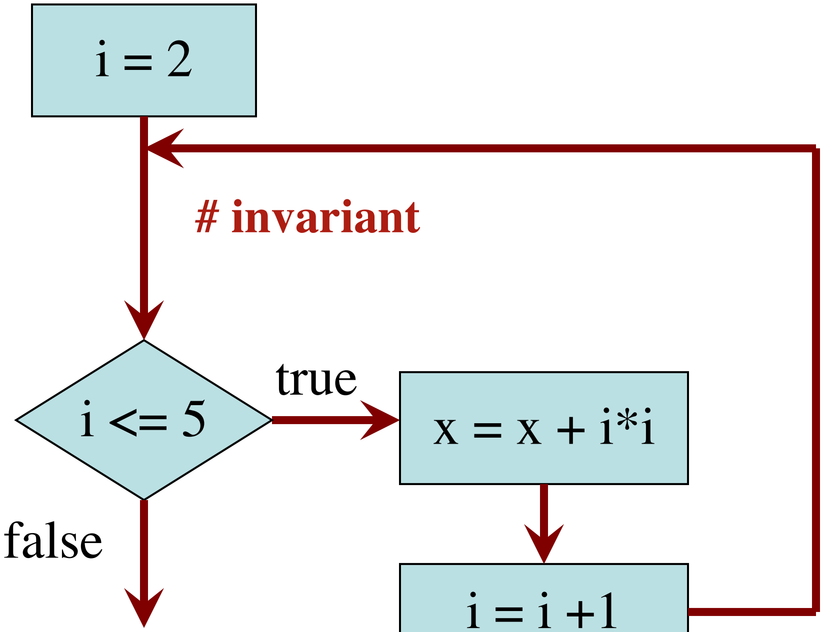
```
x = 0; i = 2
# Inv: x = sum of squares of 2..i-1
while i <= 5:
    x = x + i*i
    i = i +1
# Post: x = sum of squares of 2..5
```

Integers that have been processed:

Range 2..i-1: 2..1 (empty)

x 0

i ~~?~~ 2



i = 2

invariant

i <= 5

true

x = x + i*i

i = i +1

false

The loop processes the range 2..5

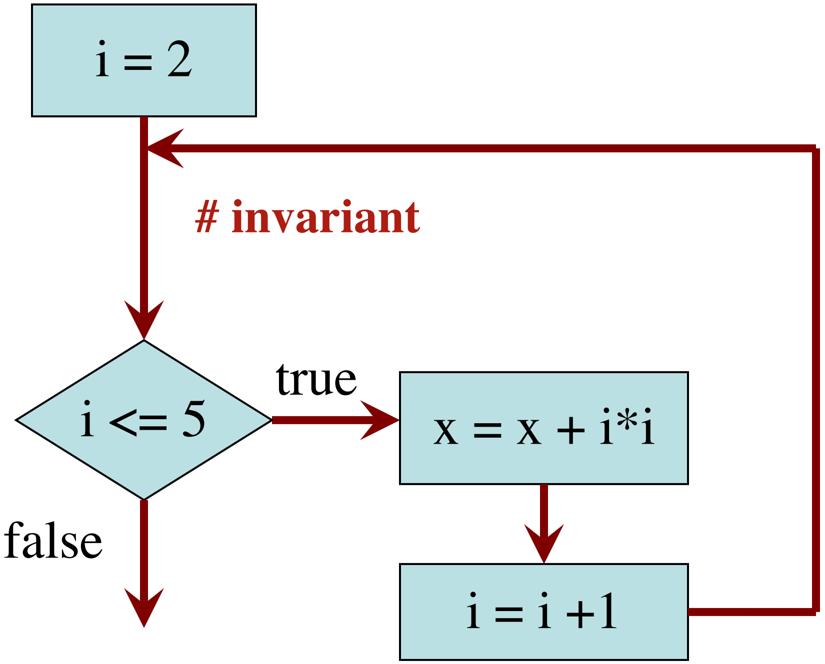
Invariants: Assertions That Do Not Change

```
x = 0; i = 2
# Inv: x = sum of squares of 2..i-1
while i <= 5:
    x = x + i*i
    i = i +1
# Post: x = sum of squares of 2..5
```

x ~~0~~ 4

i ~~?~~ ~~2~~ 3

Integers that have been processed:	2
Range 2..i-1:	2..2



The loop processes the range 2..5

Invariants: Assertions That Do Not Change

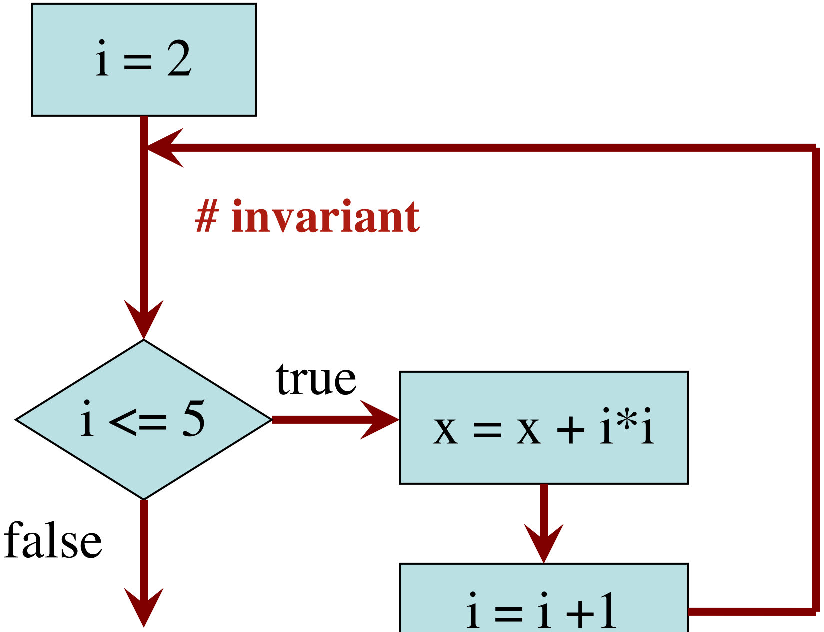
```
x = 0; i = 2
# Inv: x = sum of squares of 2..i-1
while i <= 5:
    x = x + i*i
    i = i +1
# Post: x = sum of squares of 2..5
```

Integers that have been processed: 2 , 3

Range 2..i-1: 2..3

x ~~0~~ ~~4~~ 13

i ~~?~~ ~~2~~ ~~3~~ 4



The loop processes the range 2..5

Invariants: Assertions That Do Not Change

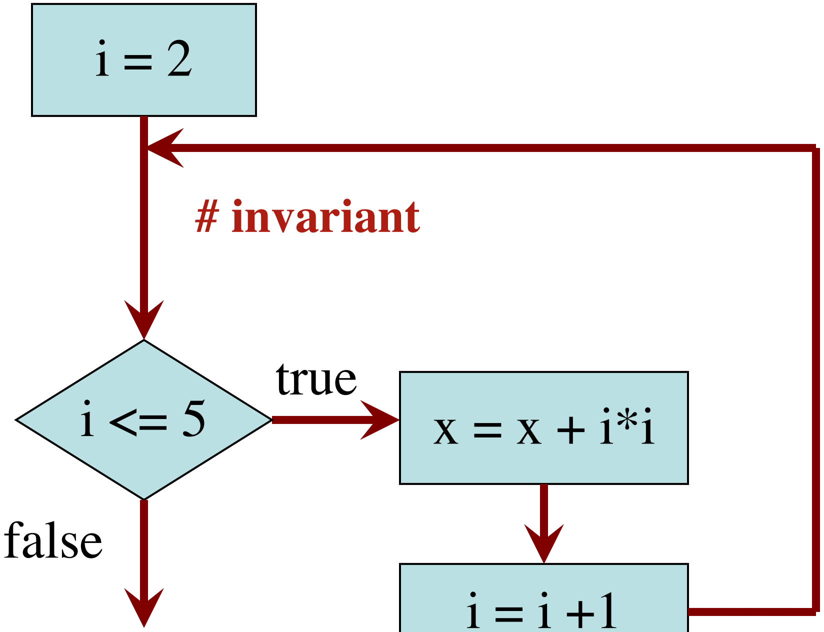
```
x = 0; i = 2
# Inv: x = sum of squares of 2..i-1
while i <= 5:
    x = x + i*i
    i = i +1
# Post: x = sum of squares of 2..5
```

Integers that have
been processed: 2, 3, 4

Range 2..i-1: 2..4

x ~~0~~ ~~4~~ ~~13~~ 29

i ~~?~~ ~~2~~ ~~3~~ ~~4~~ 5



i = 2

invariant

i <= 5

true

x = x + i*i

i = i +1

false

The loop processes the range 2..5

Invariants: Assertions That Do Not Change

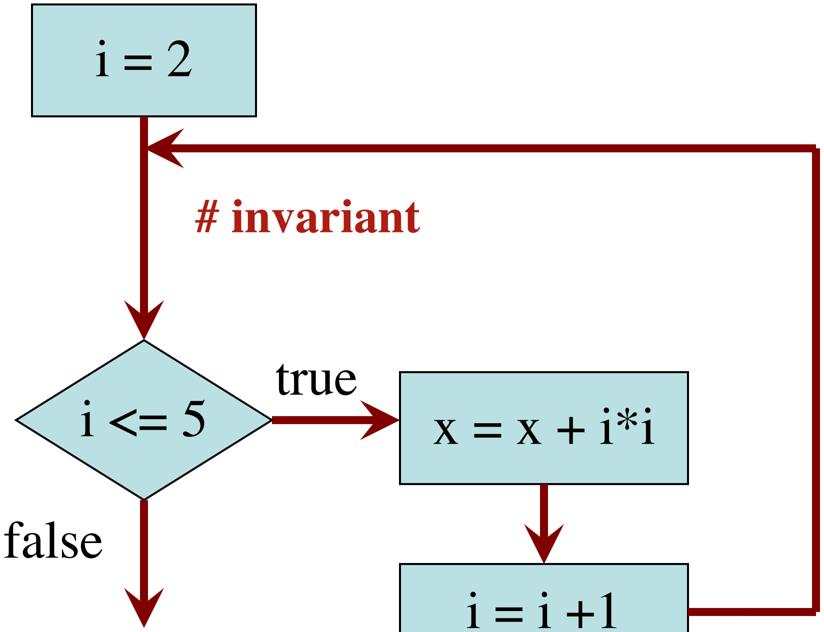
```
x = 0; i = 2
# Inv: x = sum of squares of 2..i-1
while i <= 5:
    x = x + i*i
    i = i +1
# Post: x = sum of squares of 2..5
```

Integers that have been processed: 2, 3, 4, 5

Range 2..i-1: 2..5

x ~~0~~ ~~4~~ ~~13~~ ~~29~~ 54

i ~~?~~ ~~2~~ ~~3~~ ~~4~~ ~~5~~ 6



The loop processes the range 2..5

Invariants: Assertions That Do Not Change

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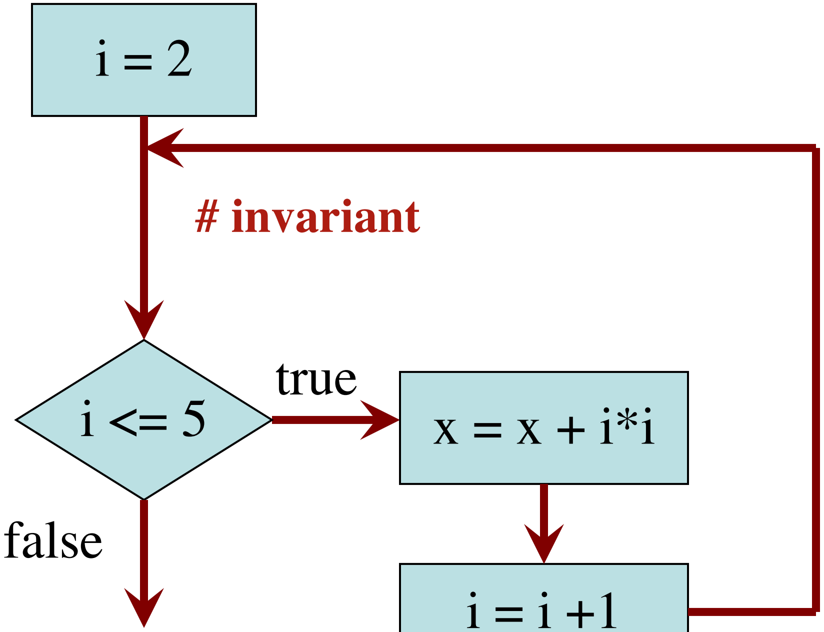
# Inv: x = sum of squares of 2..i-1
while i <= 5:
    x = x + i*i
    i = i +1
# Post: x = sum of squares of 2..5
```

x ~~0~~ ~~4~~ ~~13~~ ~~29~~ 54

i ~~?~~ ~~2~~ ~~3~~ ~~4~~ ~~5~~ 6

Integers that have been processed:	2, 3, 4, 5
Range 2..i-1:	2..5

Invariant was always true just before test of loop condition. So it's true when loop terminates



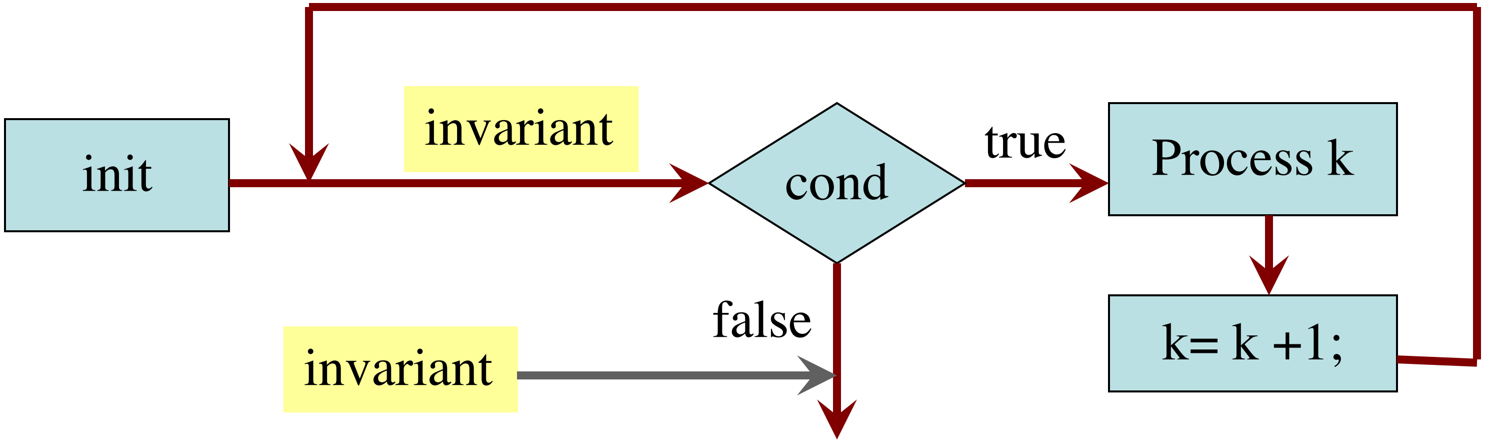
The loop processes the range 2..5

Designing Integer while-loops

```
# Process integers in a..b
# inv: integers in a..k-1 have been processed
k = a
while  k <= b:
    process integer k
    k = k + 1
# post: integers in a..b have been processed
```

Command to do something

Equivalent postcondition



Designing Integer `while`-loops

1. Recognize that a range of integers b..c has to be processed
2. Write the command and equivalent postcondition
3. Write the basic part of the while-loop
4. Write loop invariant
5. Figure out any initialization
6. Implement the repetend (process k)

Designing Integer while-loops

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```
# Process b..c










# Postcondition: range b..c has been processed
```

Designing Integer while-loops

1. Recognize that a range of integers b..c has to be processed
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```
# Process b..c



while  k <= c:

    k = k + 1
# Postcondition: range b..c has been processed
```

Designing Integer **while-loops**

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```
# Process b..c


# Invariant: range b..k-1 has been processed
while  k <= c:


    k = k + 1
# Postcondition: range b..c has been processed
```

Designing Integer while-loops

1. Recognize that a range of integers b..c has to be processed
2. Write the command and equivalent postcondition
3. Write the basic part of the while-loop
4. Write loop invariant
5. Figure out any initialization
6. Implement the repetend (process k)

```
# Process b..c
Initialize variables (if necessary) to make invariant true
# Invariant: range b..k-1 has been processed
while k <= c:
    # Process k
    k = k + 1
# Postcondition: range b..c has been processed
```

Finding an Invariant

Command to do something

```
# Make b True if n is prime, False otherwise
```

```
# b is True if no int in 2..n-1 divides n, False otherwise
```

Equivalent postcondition

What is the invariant?

Finding an Invariant

Command to do something

```
# Make b True if n is prime, False otherwise




while k < n:
    # Process k;



    k = k +1
# b is True if no int in 2..n-1 divides n, False otherwise
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Equivalent postcondition

What is the invariant?

Finding an Invariant

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# Make b True if n is prime, False otherwise



# invariant: b is True if no int in 2..k-1 divides n, False otherwise
while k < n:
    # Process k;



    k = k +1
# b is True if no int in 2..n-1 divides n, False otherwise
```

Equivalent postcondition

What is the invariant?

$1\ \underline{2\ \ 3\ \ \ldots\ \ k-1}\ \ k\ \ k+1\ \ldots\ n$

Finding an Invariant

Command to do something

```
# Make b True if n is prime, False otherwise
b = True
k = 2
# invariant: b is True if no int in 2..k-1 divides n, False otherwise
while k < n:
    # Process k;



    k = k +1
# b is True if no int in 2..n-1 divides n, False otherwise
```

Equivalent postcondition

What is the invariant?

1 2 3 … k-1 k k+1 … n

Finding an Invariant

Command to do something

```
# Make b True if n is prime, False otherwise
b = True
k = 2
# invariant: b is True if no int in 2..k-1 divides n, False otherwise
while k < n:
    # Process k;
    if n % k == 0:
        b = False
    k = k +1
# b is True if no int in 2..n-1 divides n, False otherwise
```

Equivalent postcondition

What is the invariant?

1 $\underline{2\ \ 3\ \ \ldots\ \ k\text{-}1}$ k k+1 … n

Finding an Invariant

```
# set x to # adjacent equal pairs in s

while k < len(s):
    # Process k

    k = k + 1
# x = # adjacent equal pairs in s[0..len(s)-1]
```

Command to do something

for s = 'ebeee', x = 2

Equivalent postcondition

k: next integer to process.
Which have been processed?

- A: 0..k
- B: 1..k
- C: 0..k–1
- D: 1..k–1
- E: I don’t know

Finding an Invariant

```
# set x to # adjacent equal pairs in s



while k < len(s):
    # Process k

    k = k + 1
# x = # adjacent equal pairs in s[0..len(s)-1]
```

Command to do something

for s = 'ebeee', x = 2

Equivalent postcondition

---

k: next integer to process.
Which have been processed?

- A: 0..k
- B: 1..k
- C: [✓ 0..k–1]
- D: 1..k–1
- E: I don’t know

What is the invariant?

- A: x = no. adj. equal pairs in s[1..k]
- B: x = no. adj. equal pairs in s[0..k]
- C: x = no. adj. equal pairs in s[1..k–1]
- D: x = no. adj. equal pairs in s[0..k–1]
- E: I don’t know

Finding an Invariant

```
# set x to # adjacent equal pairs in s



# inv: x = # adjacent equal pairs in s[0..k-1]
while k < len(s):
    # Process k

    k = k + 1
# x = # adjacent equal pairs in s[0..len(s)-1]
```

Command to do something

for s = 'ebeee', x = 2

Equivalent postcondition

---

k: next integer to process.
Which have been processed?

- A: 0..k
- B: 1..k
- C: [✓ 0..k–1]
- D: 1..k–1
- E: I don't know

What is the invariant?

- A: x = no. adj. equal pairs in s[1..k]
- B: x = no. adj. equal pairs in s[0..k]
- C: x = no. adj. equal pairs in s[1..k–1]
- D: [✓ x = no. adj. equal pairs in s[0..k–1]]
- E: I don't know

Finding an Invariant

```
# set x to # adjacent equal pairs in s
x = 0

# inv: x = # adjacent equal pairs in s[0..k-1]
while k < len(s):
    # Process k

    k = k + 1
# x = # adjacent equal pairs in s[0..len(s)-1]
```

Command to do something

for s = 'ebeee', x = 2

Equivalent postcondition

---

k: next integer to process.

What is initialization for k?

A: k = 0

B: k = 1

C: k = –1

D: I don’t know

Finding an Invariant

```
# set x to # adjacent equal pairs in s
x = 0
k = 1
# inv: x = # adjacent equal pairs in s[0..k-1]
while k < len(s):
    # Process k

    k = k + 1
# x = # adjacent equal pairs in s[0..len(s)-1]
```

Command to do something

for s = 'ebeee', x = 2

Equivalent postcondition

k: next integer to process.
What is initialization for k?

- A: k = 0
- B: [✓ k = 1]
- C: k = –1
- D: I don’t know

Which do we compare to “process” k?

- A: s[k] and s[k+1]
- B: s[k-1] and s[k]
- C: s[k-1] and s[k+1]
- D: s[k] and s[n]
- E: I don’t know

Finding an Invariant

```
# set x to # adjacent equal pairs in s
x = 0
k = 1
# inv: x = # adjacent equal pairs in s[0..k-1]
while k < len(s):
    # Process k
    x = x + 1 if (s[k-1] == s[k]) else 0
    k = k + 1
# x = # adjacent equal pairs in s[0..len(s)-1]
```

Command to do something

for s = 'ebeee', x = 2

Equivalent postcondition

---

k: next integer to process.
What is initialization for k?

A: k = 0
B: [✓ k = 1]
C: k = –1
D: I don’t know

Which do we compare to “process” k?

A: s[k] and s[k+1]
B: [✓ s[k-1] and s[k]]
C: s[k-1] and s[k+1]
D: s[k] and s[n]
E: I don’t know

Reason carefully about initialization

```
# s is a string; len(s) >= 1
# Set c to largest element in s
c = ??
k = ??
# inv:
while k < len(s):
    # Process k
    k = k+1
#  c = largest char in s[0..len(s)−1]
```

Command to do something

Equivalent postcondition

1. What is the invariant?

Reason carefully about initialization

```
# s is a string; len(s) >= 1
# Set c to largest element in s
c = ??
k = ??
# inv: c is largest element in s[0..k–1]
while k < len(s):
    # Process k
    k = k+1
#  c = largest char in s[0..len(s)–1]
```

Command to do something

Equivalent postcondition

1. What is the invariant?

Reason carefully about initialization

```
# s is a string; len(s) >= 1
# Set c to largest element in s
c = ??
k = ??
# inv: c is largest element in s[0..k–1]
while k < len(s):
    # Process k
    k = k+1
#  c = largest char in s[0..len(s)–1]
```

Command to do something

Equivalent postcondition

1. What is the invariant?
2. How do we initialize c and k?

A: `k = 0;  c = s[0]`

B: `k = 1;  c = s[0]`

C: `k = 1;  c = s[1]`

D: `k = 0;  c = s[1]`

E: None of the above

Reason carefully about initialization

```
# s is a string; len(s) >= 1
# Set c to largest element in s
c = ??
k = ??
# inv: c is largest element in s[0..k–1]
while k < len(s):
    # Process k
    k = k+1
# c = largest char in s[0..len(s)–1]
```

Command to do something

Equivalent postcondition

1. What is the invariant?
2. How do we initialize c and k?

A: `k = 0; c = s[0]`

B: `k = 1; c = s[0]`

C: `k = 1; c = s[1]`

D: `k = 0; c = s[1]`

E: None of the above

An empty set of characters or integers has no maximum. Therefore, be sure that 0..k–1 is not empty. You must start with k = 1.