Lecture 15

# Recursion

# Announcements for Today

## Prelim 1

- Tonight at 7:30-9pm
  - **A–J** (Uris G01)
  - **K-Z** (Statler Auditorium)
- Graded by noon on Sun
  - Scores will be in CMS
  - In time for drop date
- Make-ups were e-mailed
  - If not, e-mail Amy NOW

## Other Announcements

- Reading: 5.8 – 5.10
- Assignment 3 now graded
  - **Mean** 95, **Median** 98
  - **Time**: 8 hrs, **StdDev**: 3.5 hrs
  - Very similar to last year
- Assignment 4 posted Friday
  - Parts 1-3: Can do already
  - Part 4: material from today
  - Due two weeks from today

# Recursion

- **Recursive Definition**:
  A definition that is defined in terms of itself
- **Recursive Function**:
  A function that calls itself (directly or indirectly)

**PIP** stands for "**PIP** Installs Packages"

# A Mathematical Example: Factorial

- Non-recursive definition:

  $n! = n \times n\text{-}1 \times \ldots \times 2 \times 1$

  $\quad = n\,(n\text{-}1 \times \ldots \times 2 \times 1)$

- Recursive definition:

  $n! = n\,(n\text{-}1)!$ for $n \geq 0$ **Recursive case**

  $0! = 1$ **Base case**

What happens if there is no base case?

# Factorial as a Recursive Function

```
def factorial(n):
    """Returns: factorial of n.
    Pre: n ≥ 0 an int"""
    if n == 0:
        return 1

    return n*factorial(n-1)
```

- $n! = n\ (n-1)!$
- $0! = 1$

**Base case(s)**

**Recursive case**

What happens if there is no base case?

# Example: Fibonnaci Sequence

- Sequence of numbers: 1, 1, 2, 3, 5, 8, 13, ...

  $a_0$ $a_1$ $a_2$ $a_3$ $a_4$ $a_5$ $a_6$

  - Get the next number by adding previous two
  - What is $a_8$?

A: $a_8 = 21$
B: $a_8 = 29$
C: $a_8 = 34$
D: None of these.

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A: $a_8 = 21$
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C: $a_8 = 34$ **correct**
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  $a_0$ $a_1$ $a_2$ $a_3$ $a_4$ $a_5$ $a_6$

  - Get the next number by adding previous two
  - What is $a_8$?
- Recursive definition:
  - $a_n = a_{n\text{-}1} + a_{n\text{-}2}$ **Recursive Case**
  - $a_0 = 1$ **Base Case**
  - $a_1 = 1$ **(another) Base Case**

Why did we need two base cases this time?

# Fibonacci as a Recursive Function

```
def fibonacci(n):
    """Returns: Fibonacci no. a_n
    Precondition: n ≥ 0 an int"""
    if n <= 1:
        return 1

    return (fibonacci(n-1)+
            fibonacci(n-2))
```

**Base case(s)**

**Recursive case**

Note difference with base case conditional.

# Fibonacci as a Recursive Function

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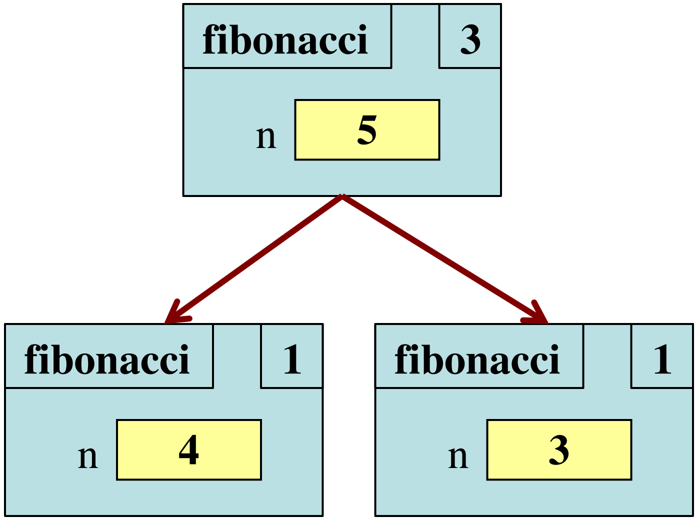
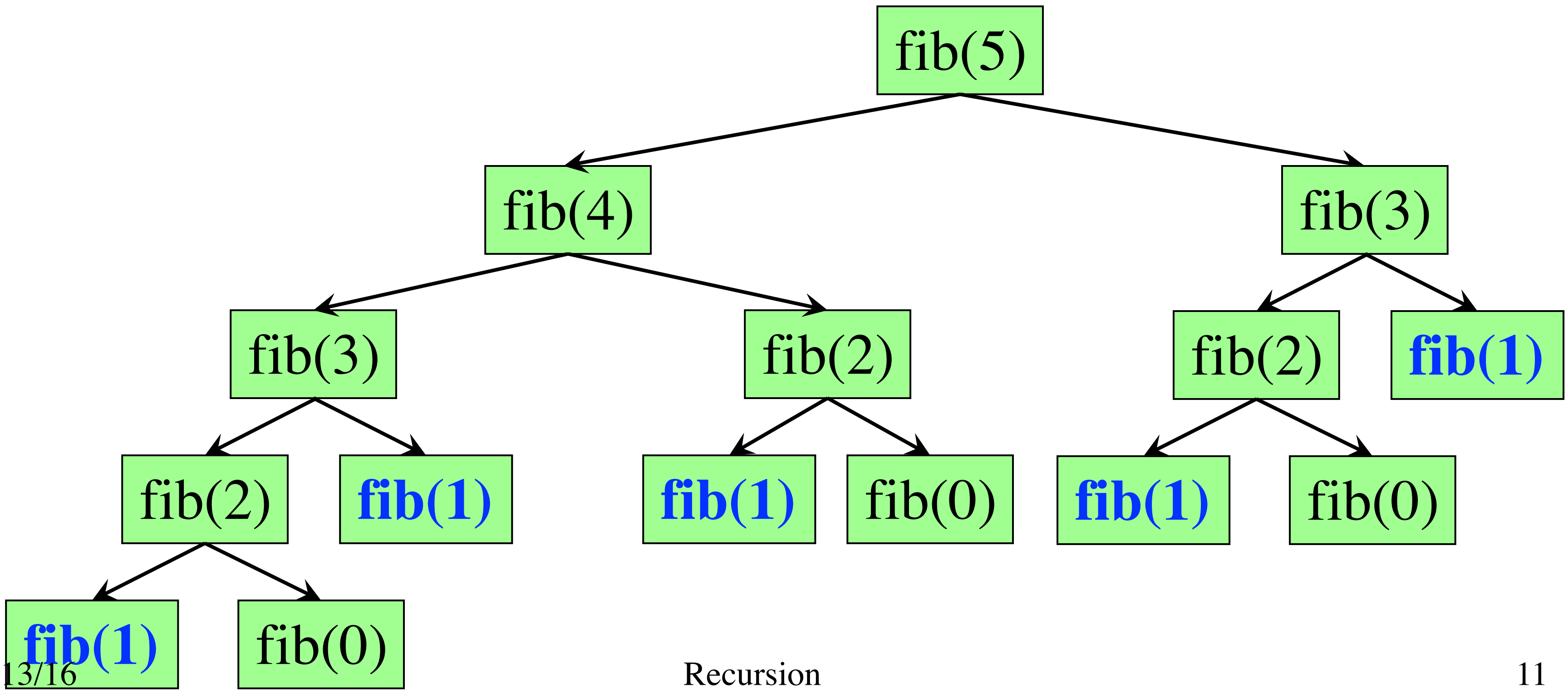
              fibonacci(n-2))
```

- Function that calls itself
  - Each call is new frame
  - Frames require memory
  - ∞ calls = ∞ memory



# Fibonacci: # of Frames vs. # of Calls

- Fibonacci is very inefficient.
  - fib($n$) has a stack that is always $\leq n$
  - But fib($n$) makes a lot of redundant calls



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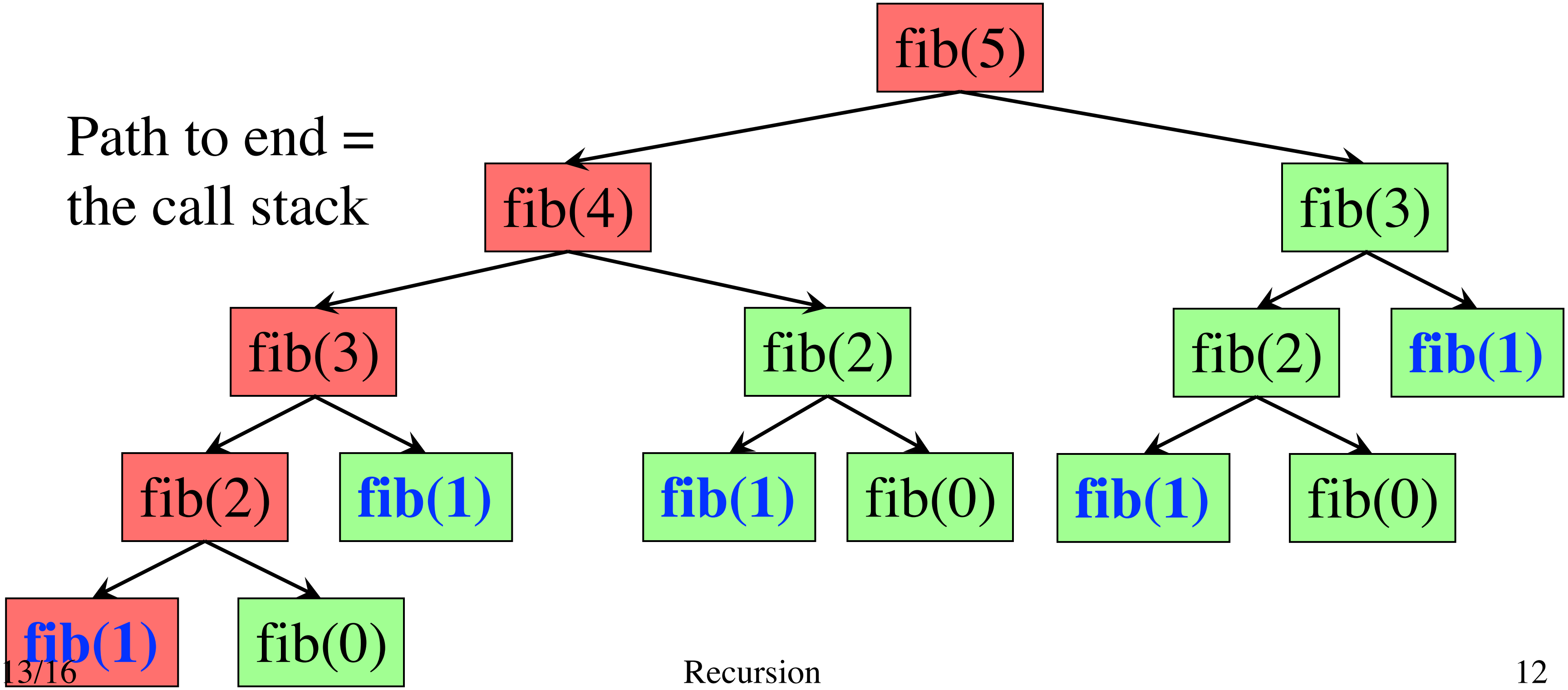
Path to end =
the call stack



# Recursion vs Iteration

- **Recursion** is ***provably equivalent*** to **iteration**
  - Iteration includes **for-loop** and **while-loop** (later)
  - Anything can do in one, can do in the other
- But some things are easier with recursion
  - And some things are easier with iteration
- Will **not** teach you when to choose recursion
  - This is a topic for more advanced classes
- We just want you to ***understand the technique***

# Recursion is best for Divide and Conquer

**Goal**: Solve problem P on a piece of data

| data |
|---|

# Recursion is best for Divide and Conquer

**Goal**: Solve problem P on a piece of data

**data**

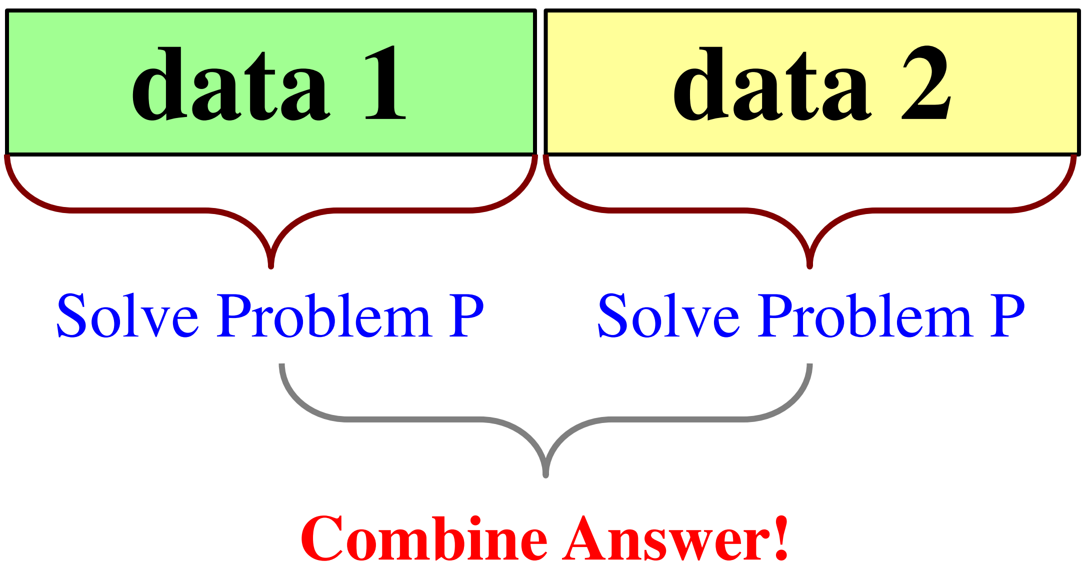
**Idea**: Split data into two parts and solve problem

| **data 1** | **data 2** |
| --- | --- |
| Solve Problem P | Solve Problem P |

# Recursion is best for Divide and Conquer

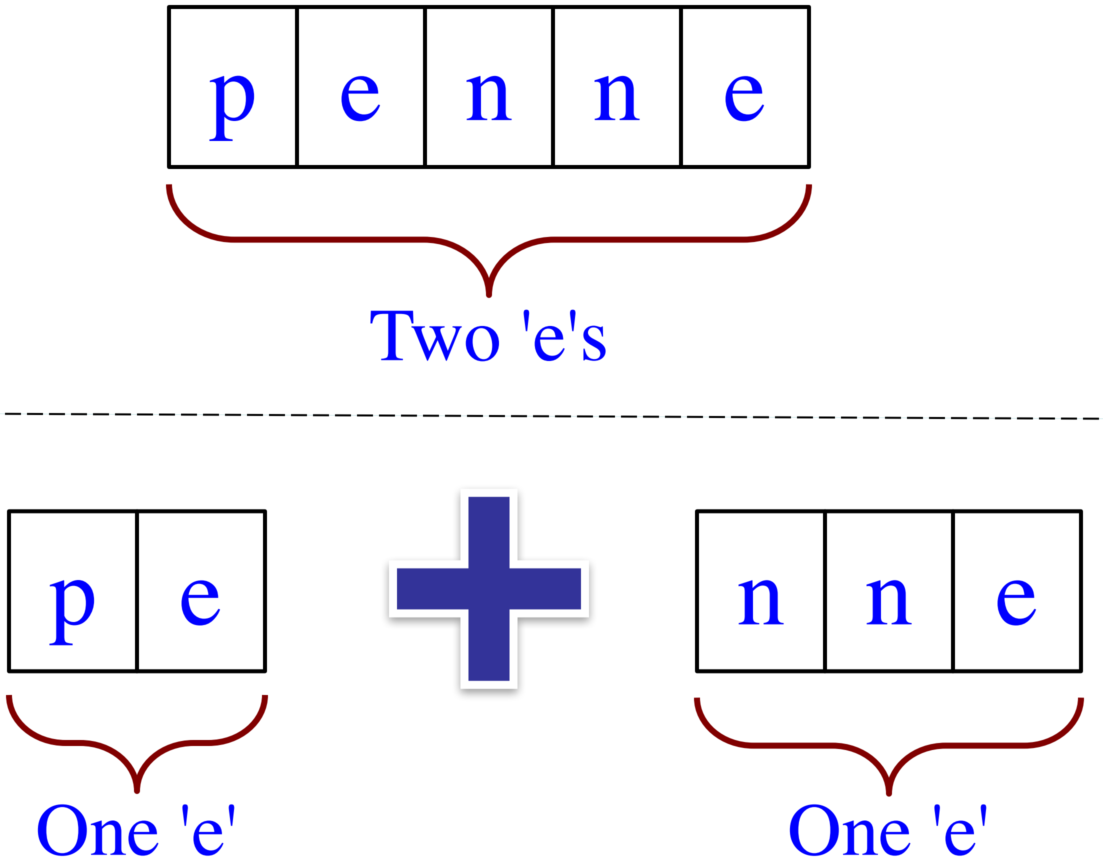
**Goal**: Solve problem P on a piece of data


**Idea**: Split data into two parts and solve problem



# Divide and Conquer Example

Count the number of 'e's in a string:



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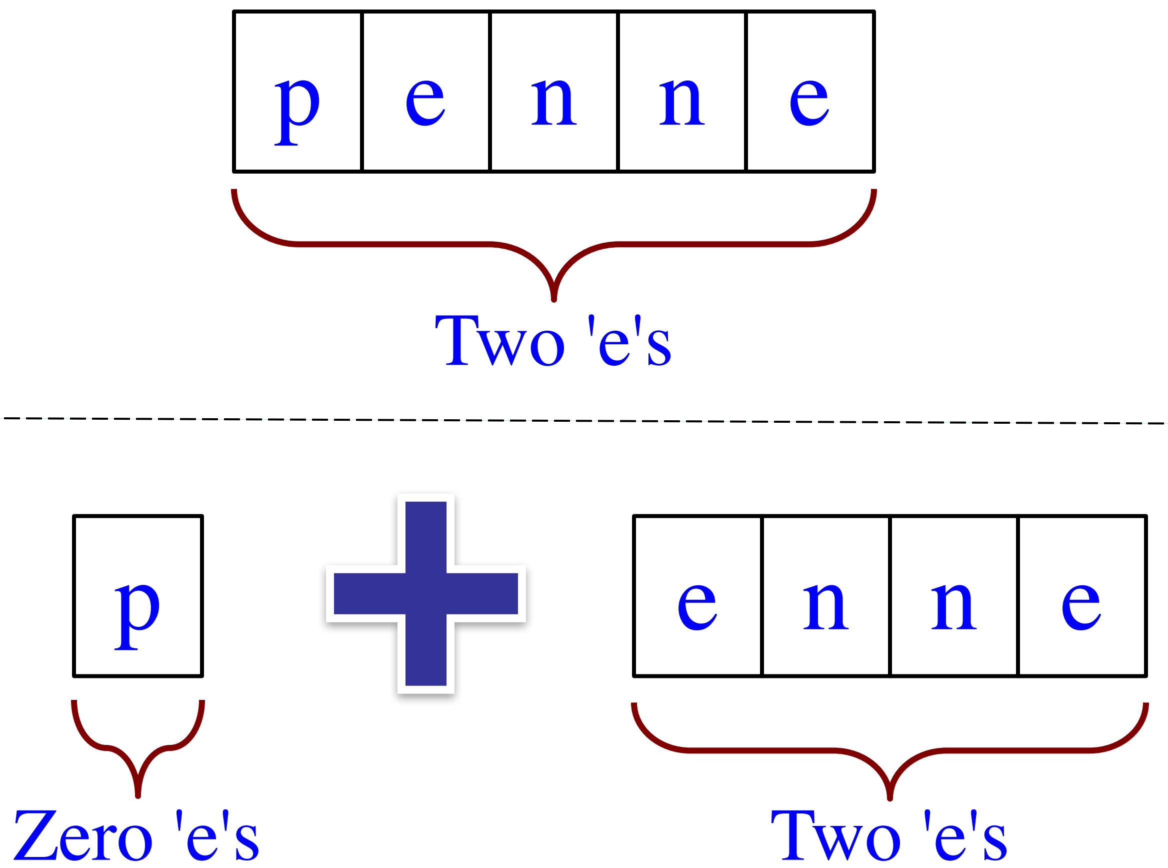

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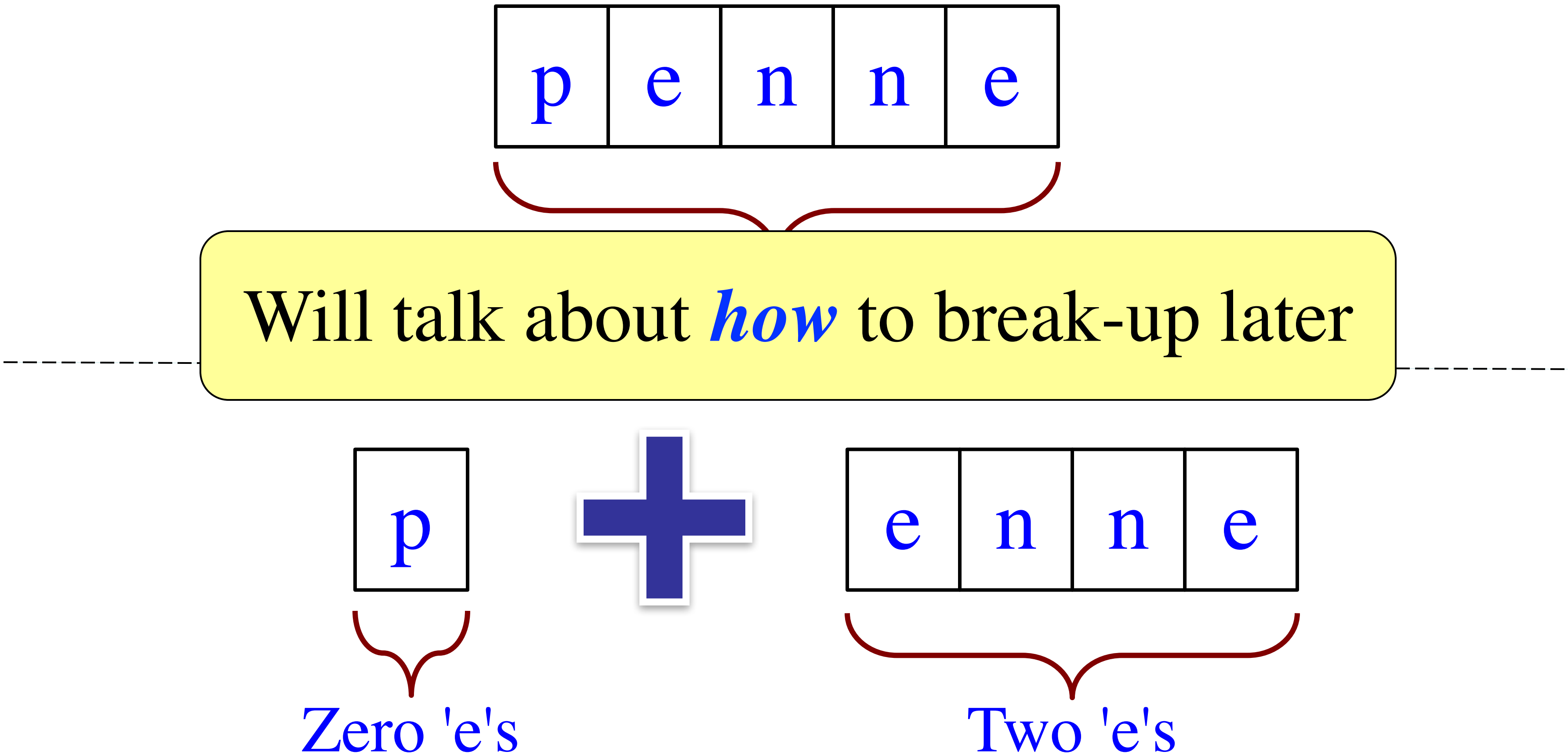

# Three Steps for Divide and Conquer

1. Decide what to do on “small” data
   - Some data cannot be broken up
   - Have to compute this answer directly
2. Decide how to break up your data
   - Both “halves” should be smaller than whole
   - Often no wrong way to do this (next lecture)
3. Decide how to combine your answers
   - Assume the smaller answers are correct
   - Combining them should give bigger answer

# Divide and Conquer Example

```
def num_es(s):
    """Returns: # of 'e's in s"""
    # 1. Handle small data
    if s == '':
        return 0
    elif len(s) == 1:
        return 1 if s[0] == 'e' else 0

    # 2. Break into two parts
    left = num_es(s[0])
    right = num_es(s[1:])

    # 3. Combine the result
    return left+right
```

"Short-cut" for

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if s[0] == 'e':
    return 1
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| s[0] | s[1:] | | | |
|---|---|---|---|---|
| p | e | n | n | e |

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```

Base Case

Recursive Case

# Exercise: Remove Blanks from a String

```
def deblank(s):
    """Returns: s but with its blanks removed"""
```

1. Decide what to do on “small” data
   - If it is the **empty string**, nothing to do

     ```
     if s == '':
         return s
     ```

   - If it is a **single character**, delete it if a blank

     ```
     if s == ' ':     # There is a space here
         return ''  # Empty string
     else:
         return s
     ```

# Exercise: Remove Blanks from a String

```
def deblank(s):
    """Returns: s but with its blanks removed"""
```

2. Decide how to break it up

```
left = deblank(s[0])      # A string with no blanks
right = deblank(s[1:])    # A string with no blanks
```

3. Decide how to combine the answer

```
return left+right         # String concatenation
```

# Putting it All Together

```
def deblank(s):
    """Returns: s w/o blanks"""
    if s == '':
        return s
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    return left+right
```

Handle small data

Break up the data

Combine answers

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Base Case

Recursive Case

# Minor Optimization

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    return left+right
```

Needed second base case to handle s[0]

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    if s[0] == ' ':
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    return left+right
```

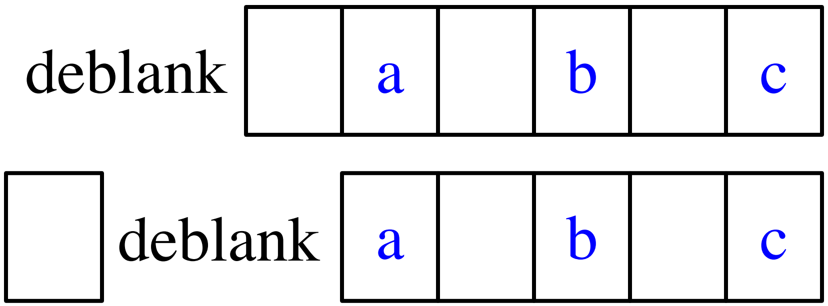
Eliminate the second base by combining

Less recursive calls

# Following the Recursion

| | | | | | | |
|---|---|---|---|---|---|---|
| deblank | | a | | b | | c |

# Following the Recursion



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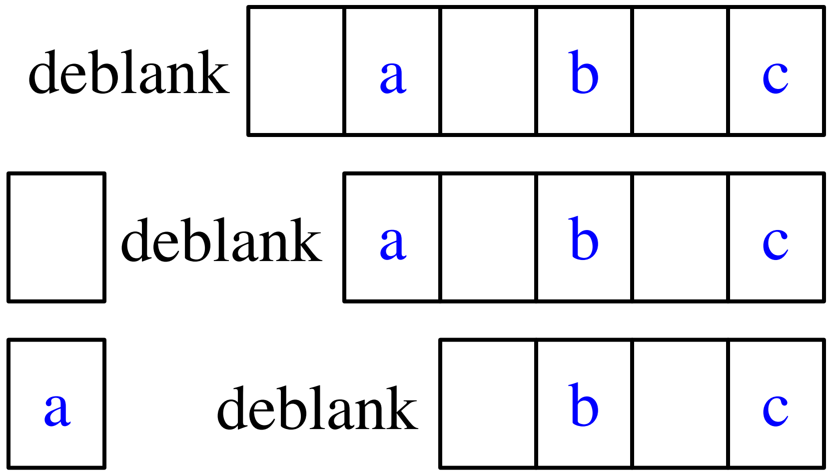

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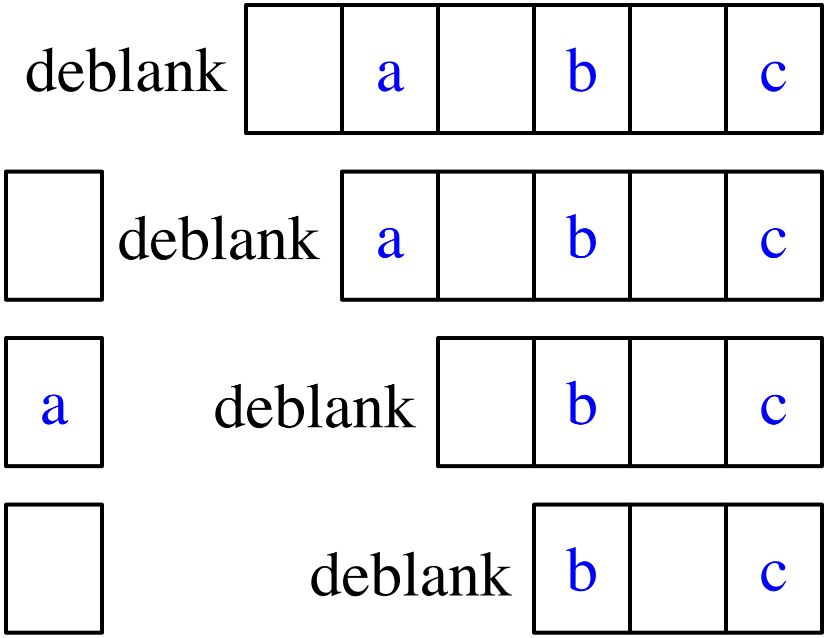

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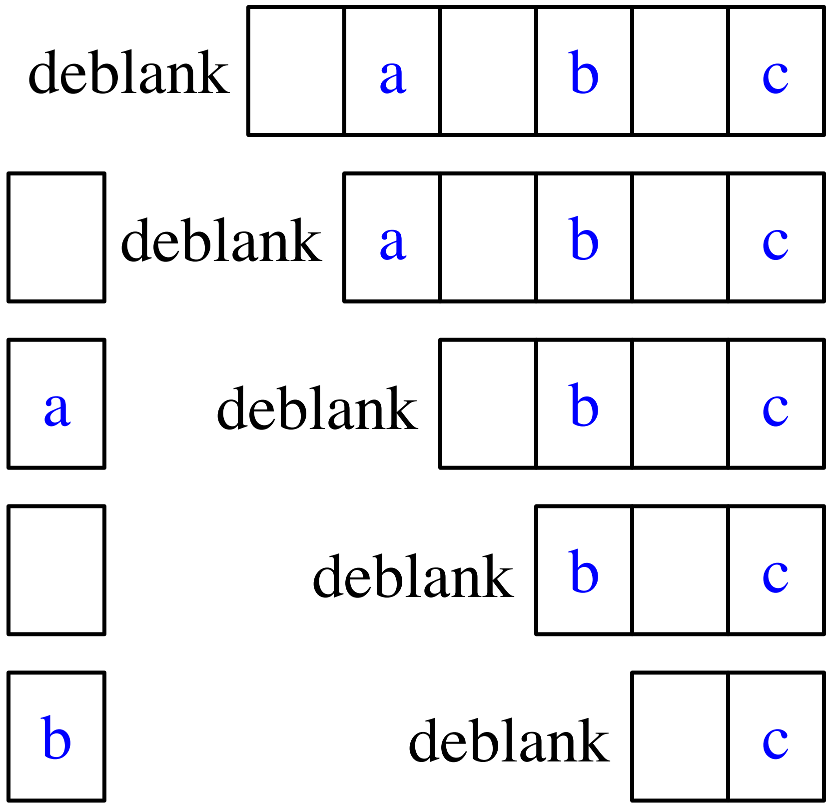

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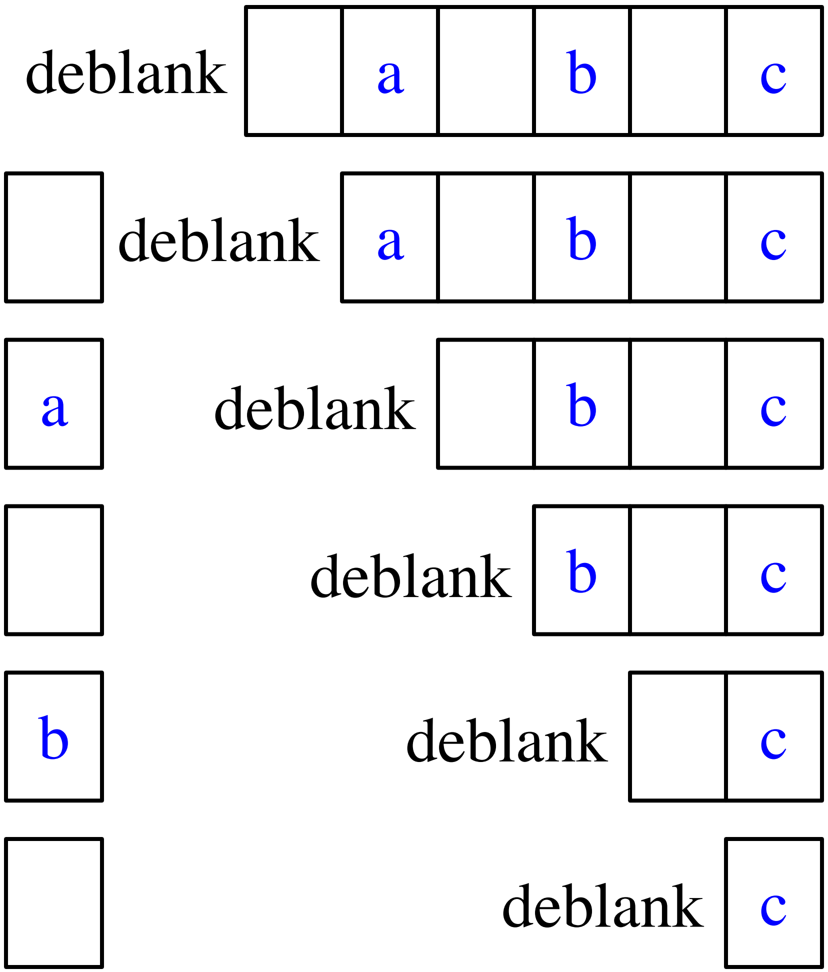

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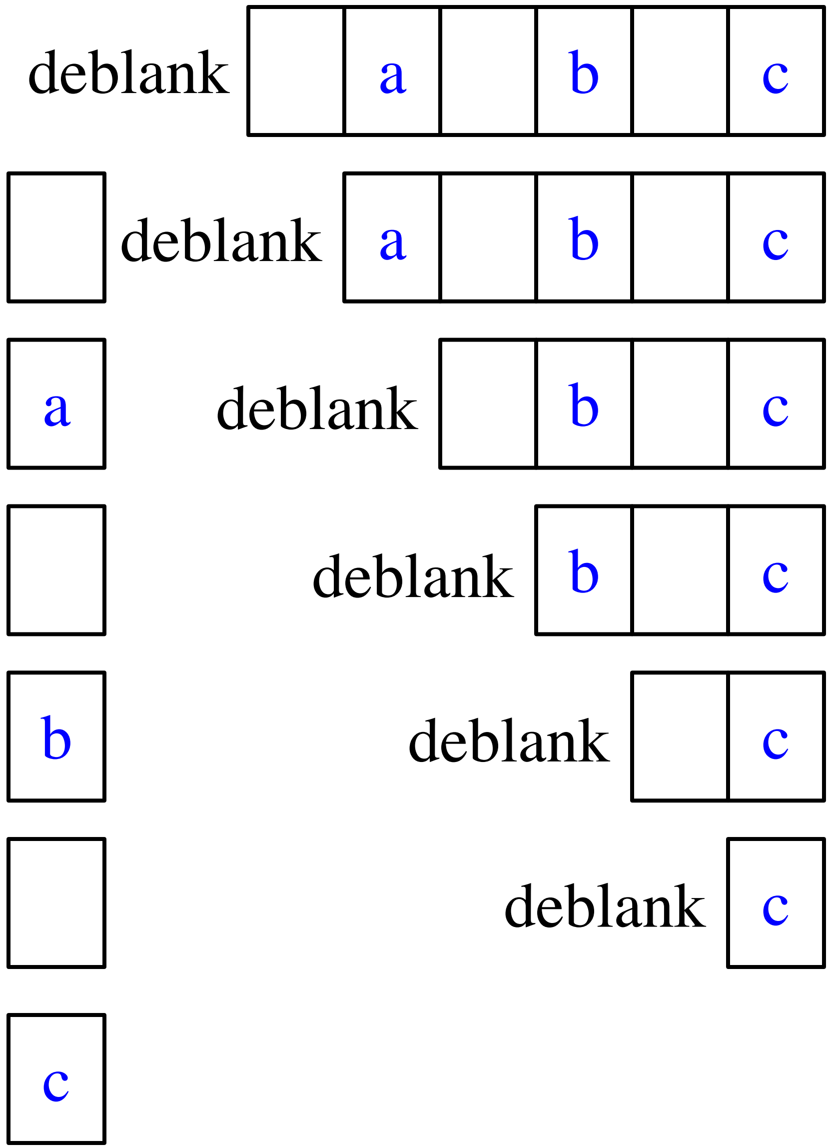

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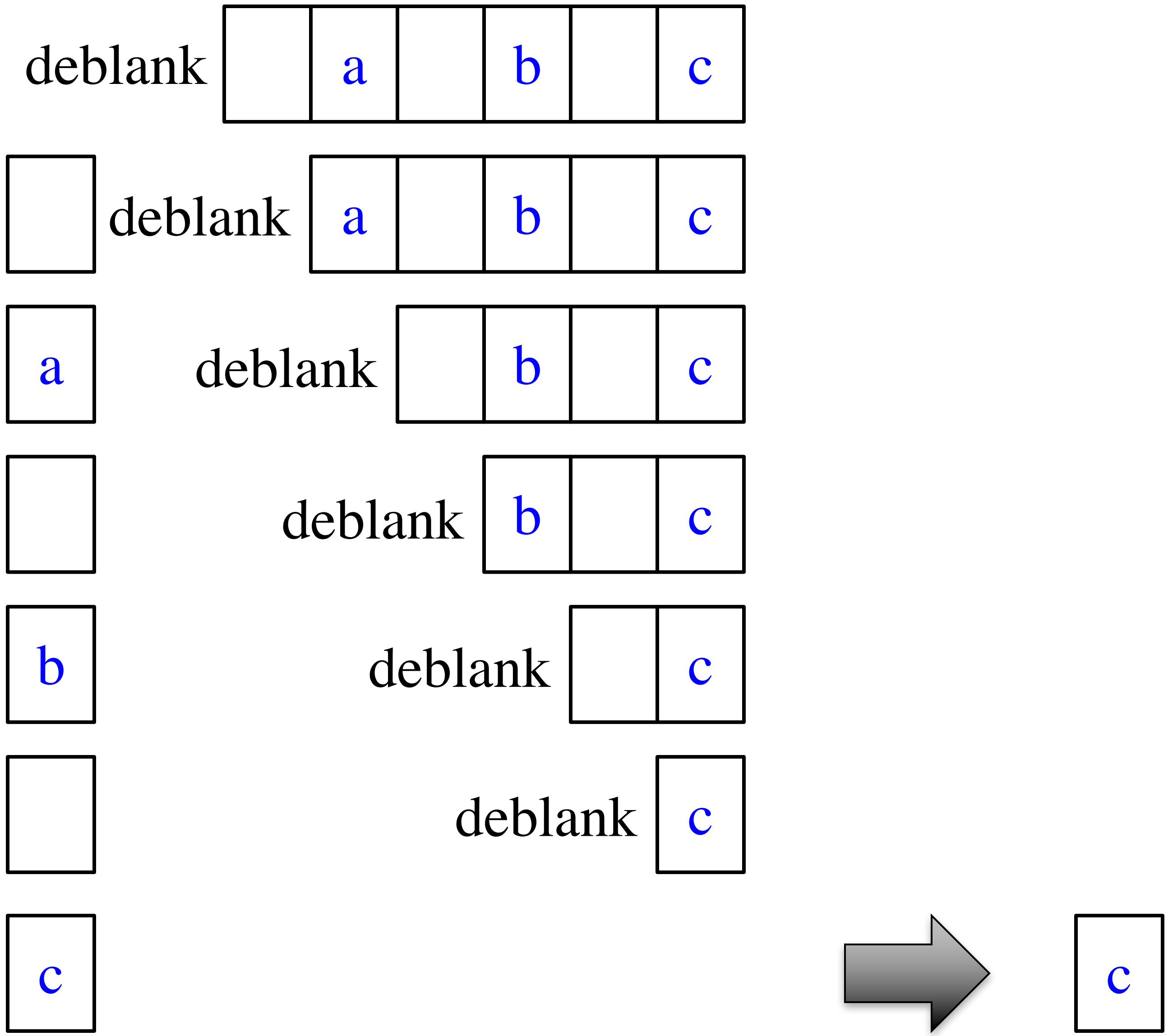

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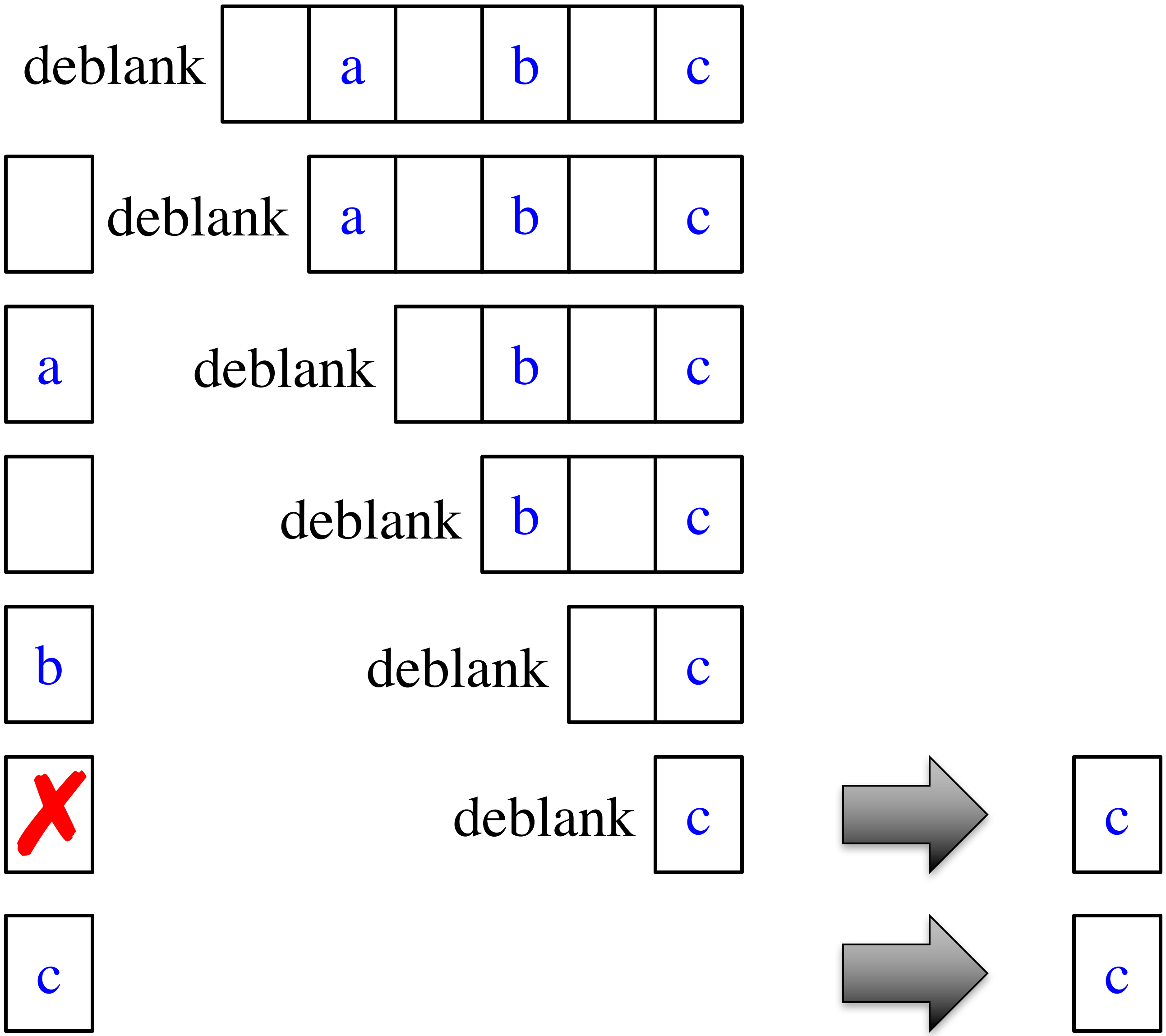

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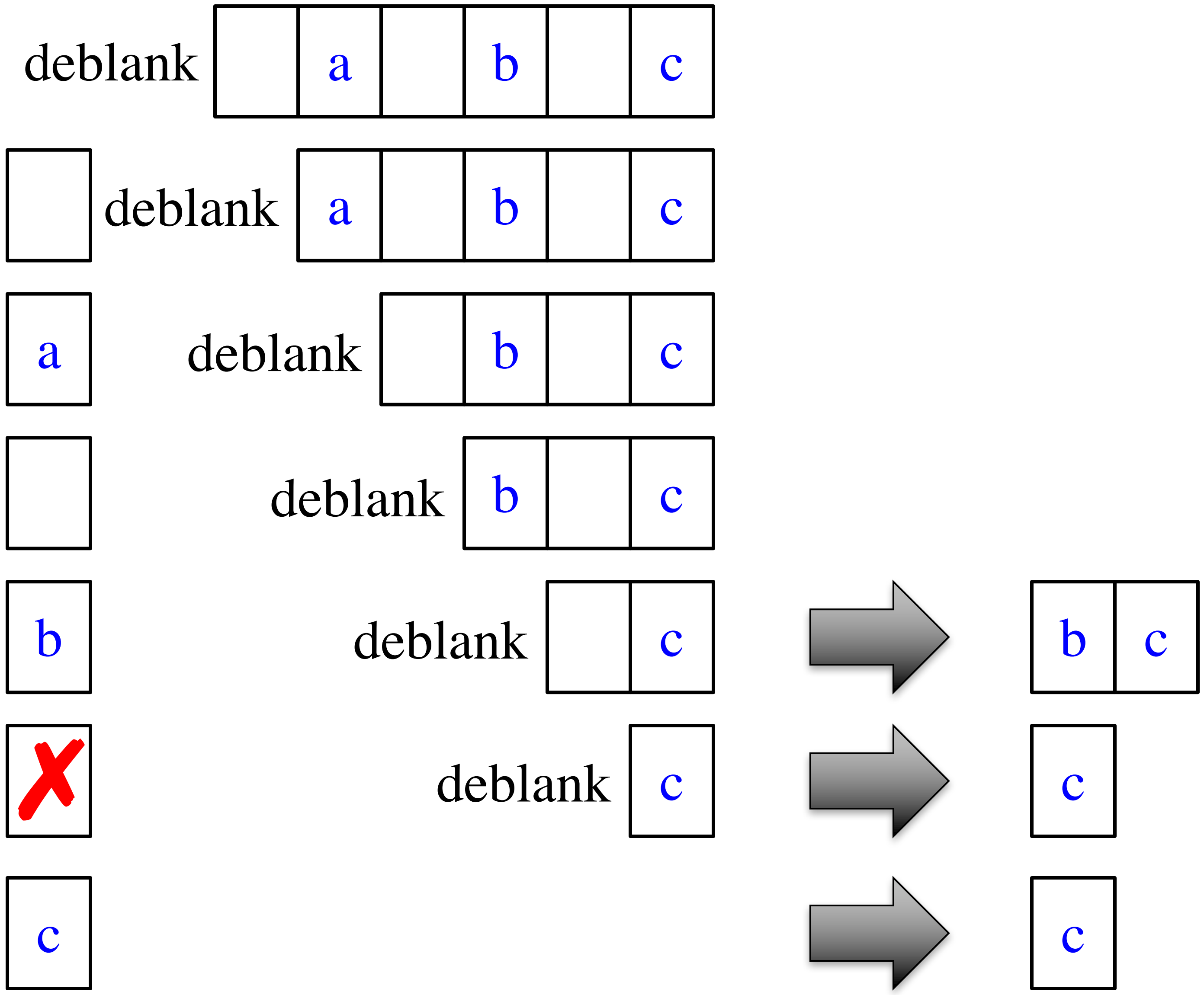

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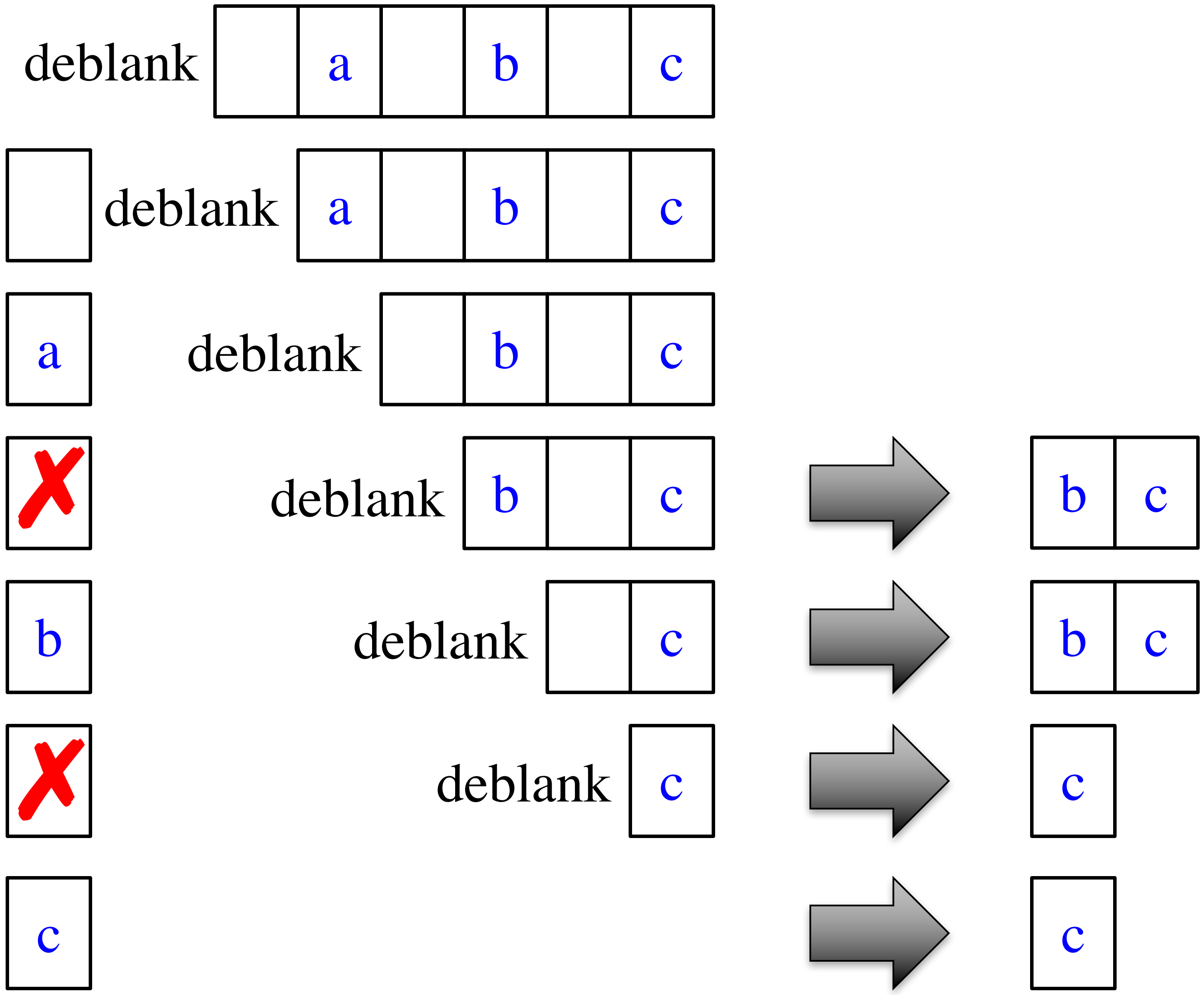

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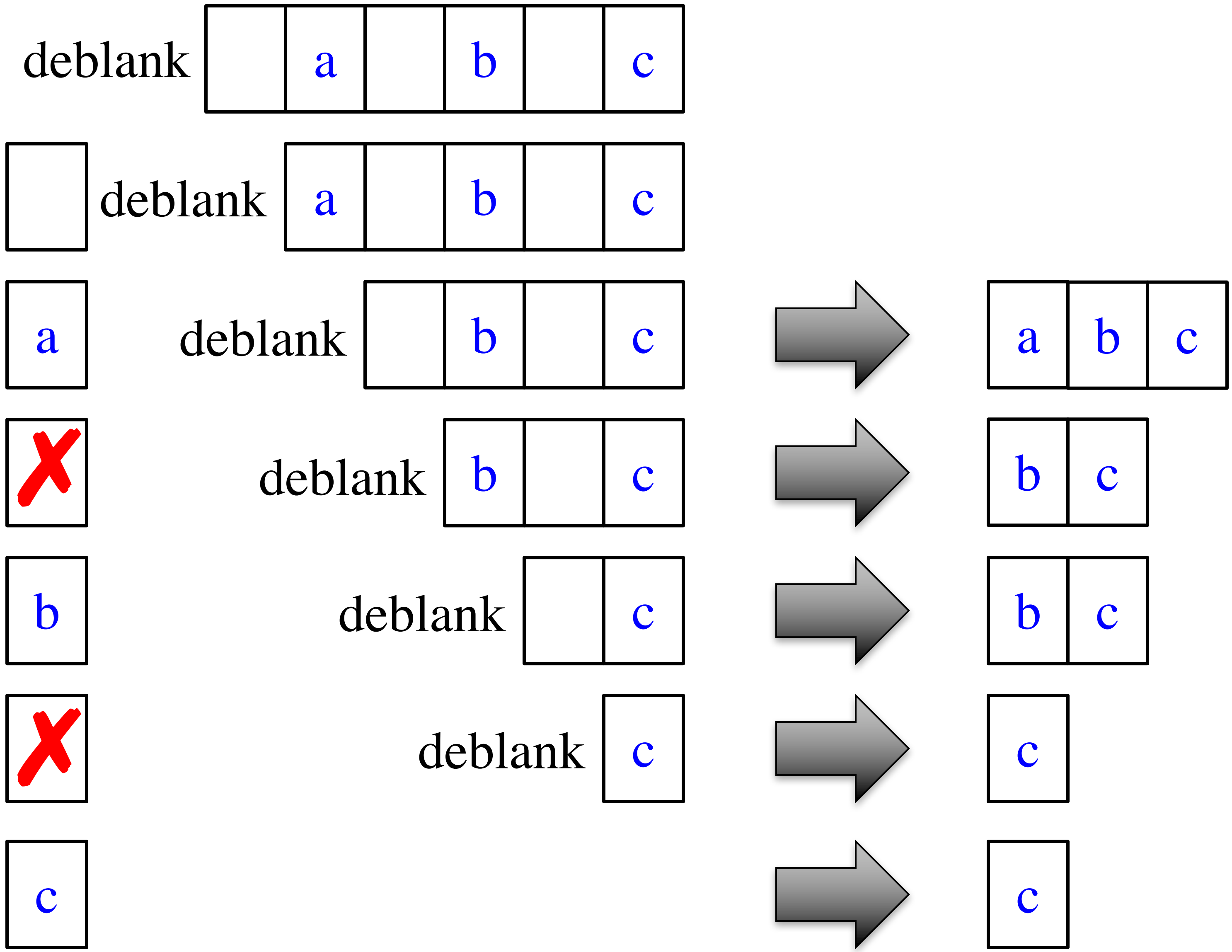

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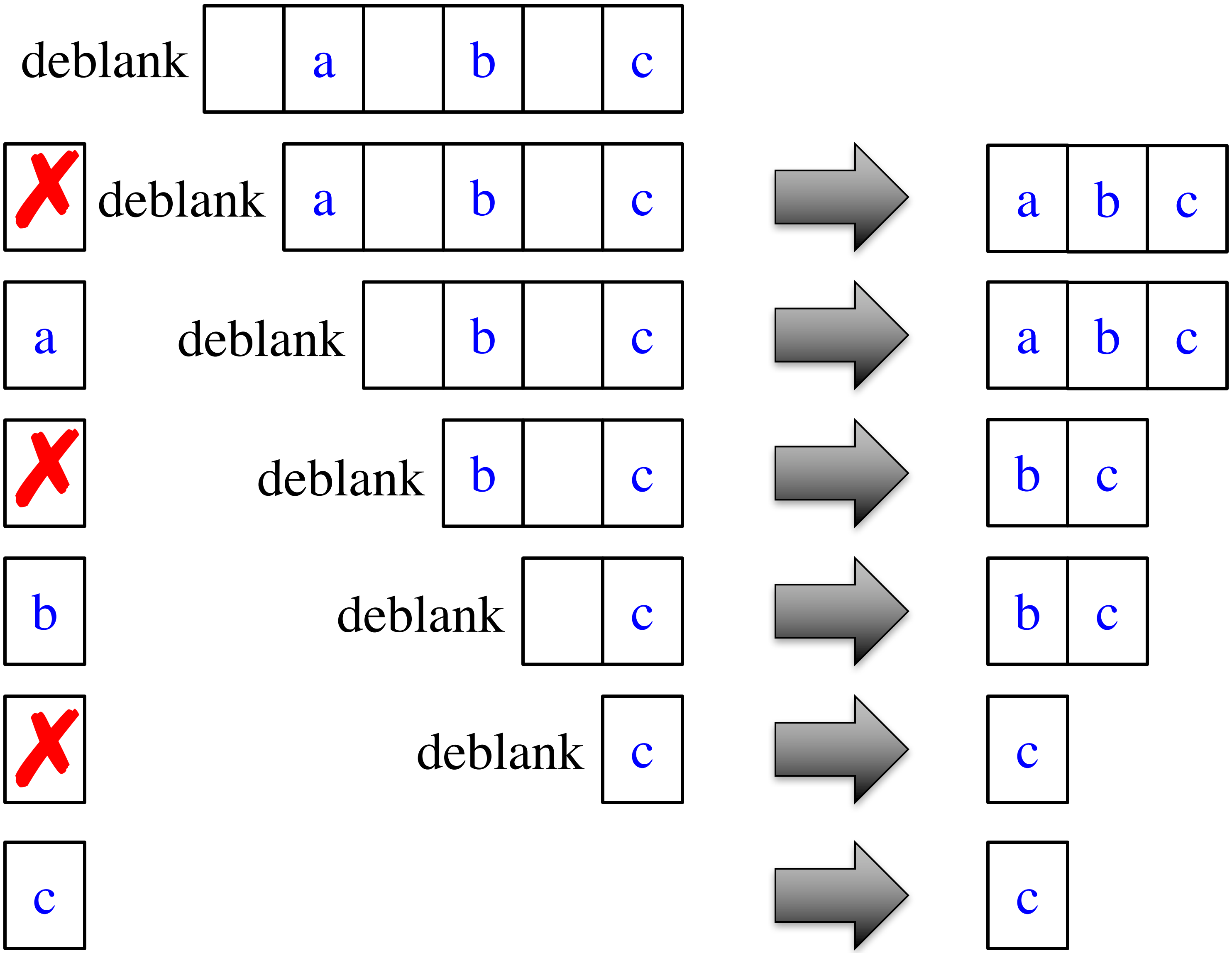

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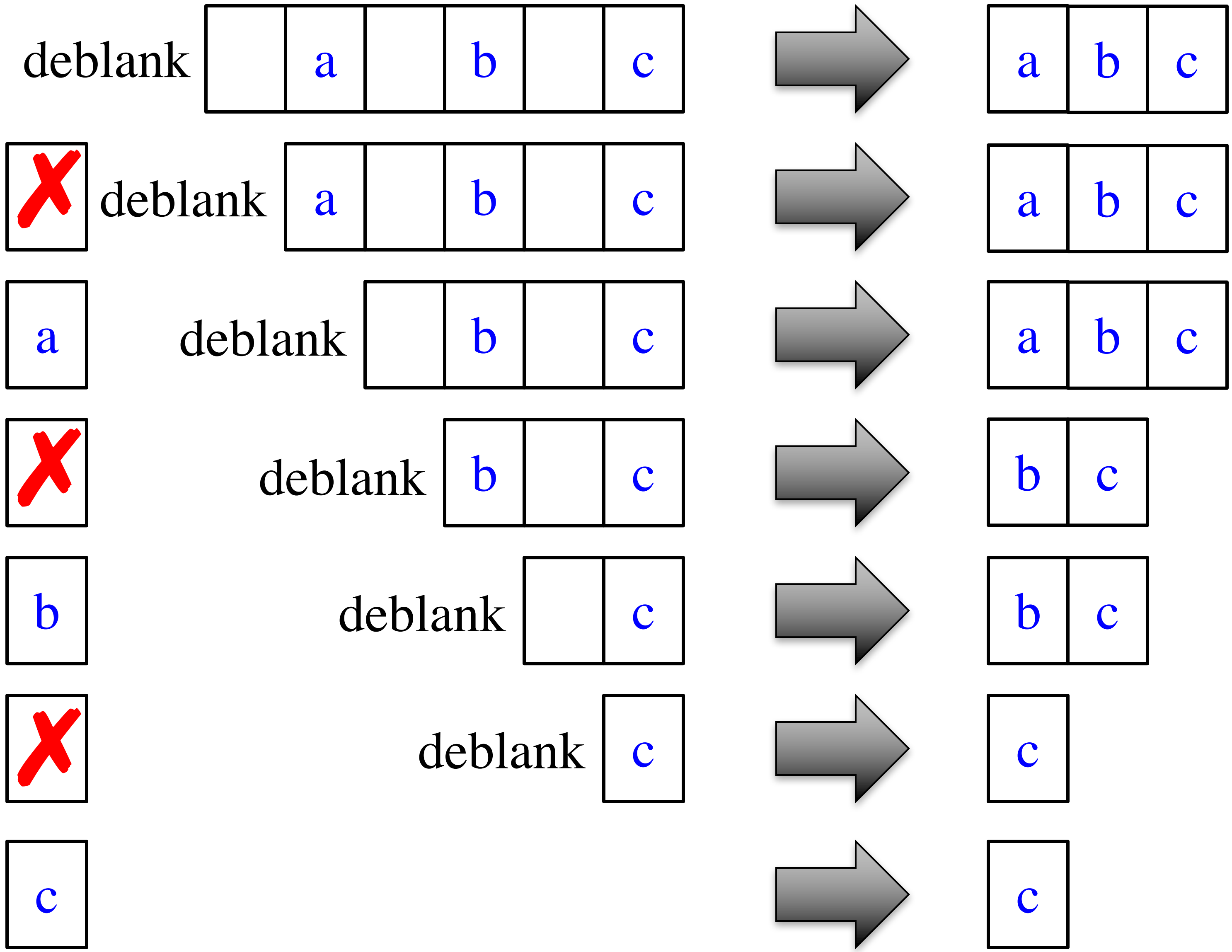

# Final Modification

```
def deblank(s):
    """Returns: s w/o blanks"""
    if s == '':
        return s

    left = s[0]
    if s[0] == ' ':
        left = ''
    right = deblank(s[1:])

    return left+right
```

Real work done here

# Final Modification

```
def deblank(s):
    """Returns: s w/o blanks"""
    if s == '':
        return s

    left = s
    if s[0] in string.whitespace
        left = ''
    right = deblank(s[1:])

    return left+right
```

Real work done here

Module string has special constants to simplify detection of whitespace and other characters.

# Next Time: Breaking Up Recursion