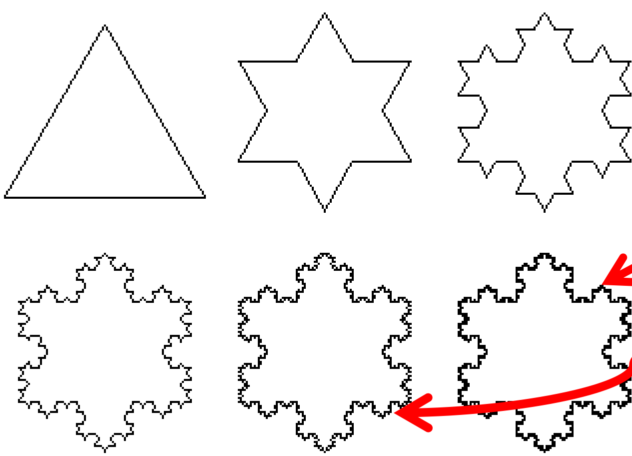
17. Recursion

Recursive Tiling
Random Mondrian
Recursive Evaluation of n!
Tracking a Recursive Function Call

What is Recursion?

A function is recursive if it calls itself.

A pattern is recursive if it is defined in terms of itself.



I can tell you what this
is in terms of what that
is.

The Concept of Recursion Is Hard But VERY Important

Teaching Plan:

Develop a recursive triangle-tiling procedure informally.

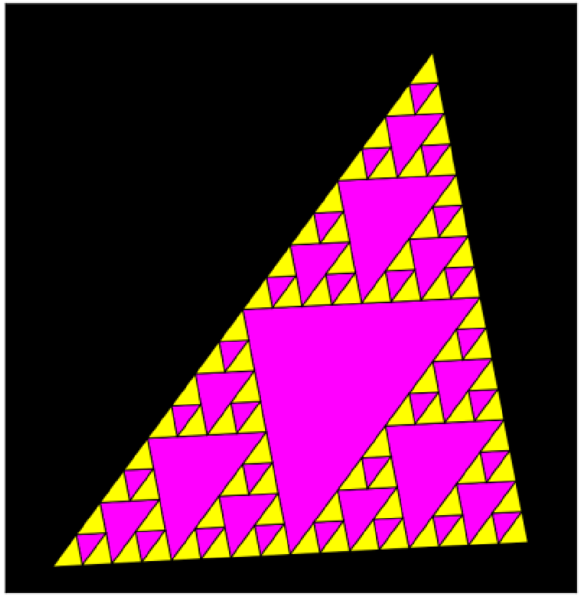
Fully implement (in Python) a recursive rectangle-tiling procedure.

Fully implement a recursive function for n!

Fully implement a recursive function for sorting (in a later lecture).

Recursive Graphics

We will develop a graphics procedure that draws this:

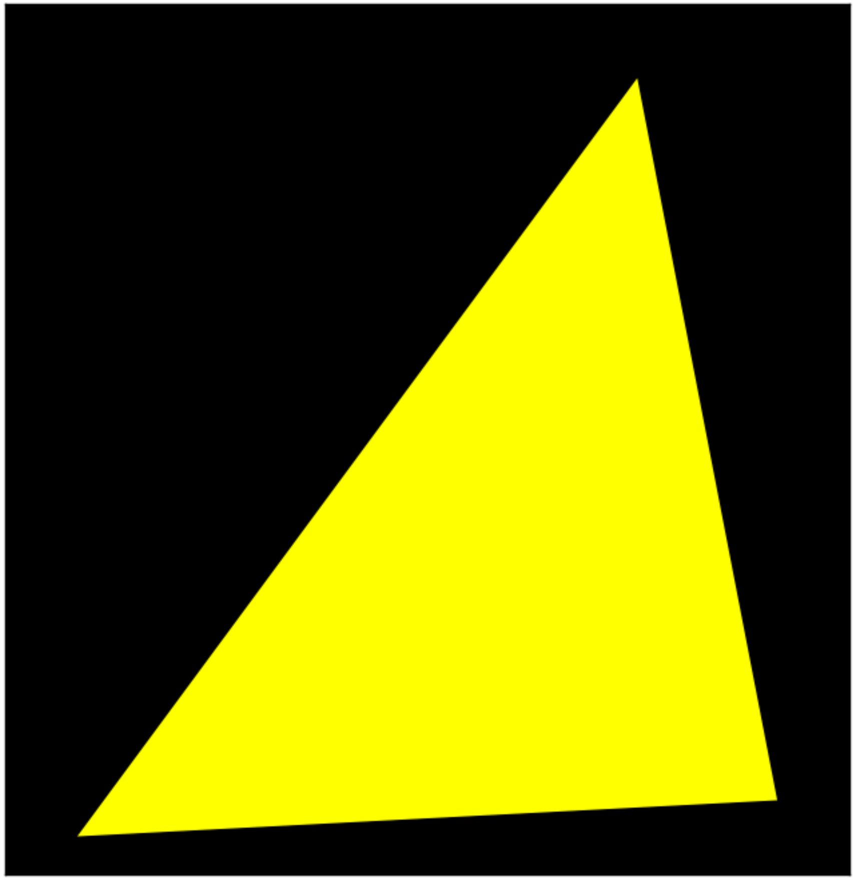


We are tiling a triangle with increasingly smaller triangles.

The procedure will call itself.

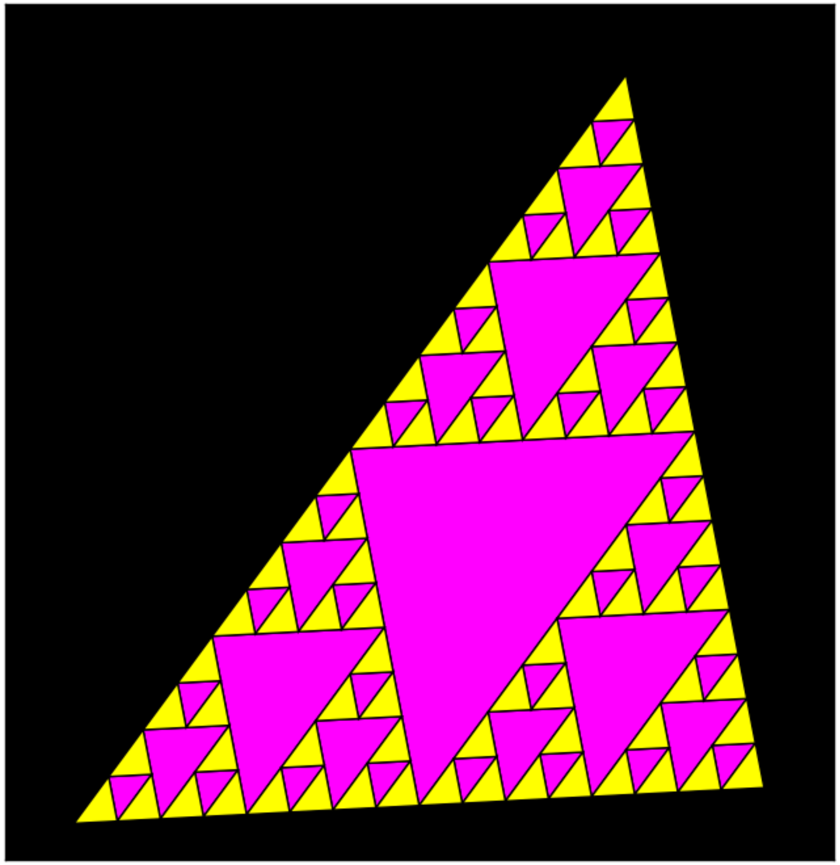
Tiling a Triangle

We start with one big triangle:

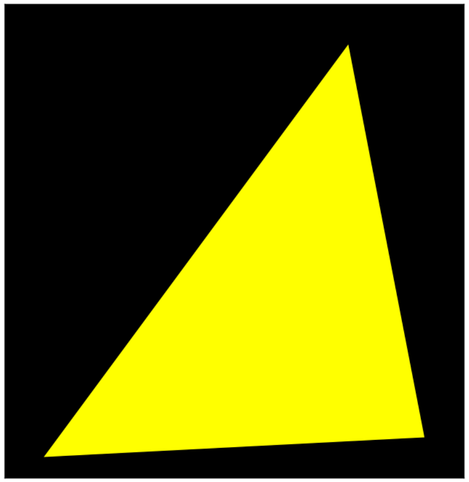


Tiling a Triangle

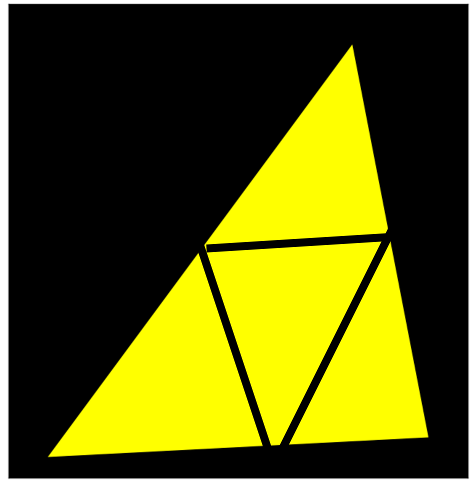
And are to end up with this:



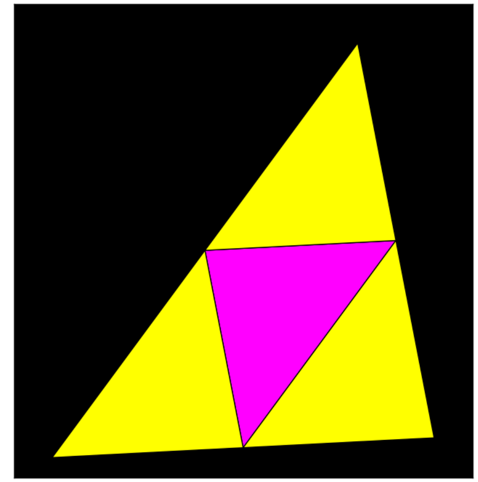
Requires Repetition



Given a yellow triangle

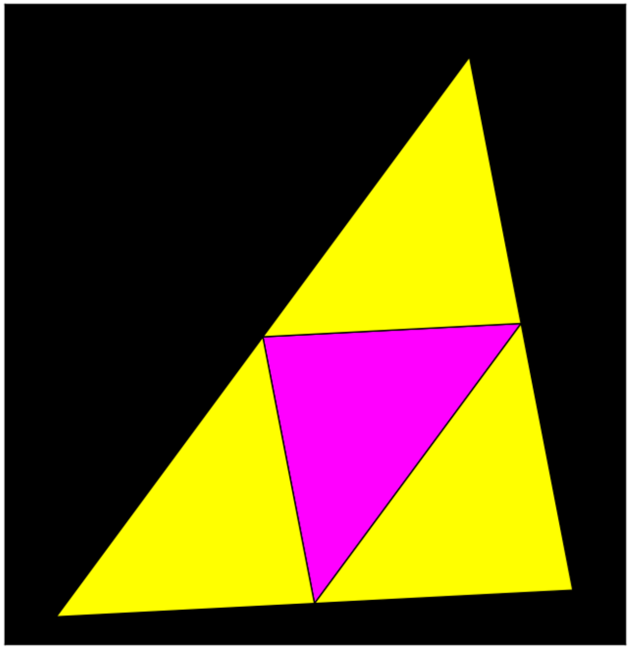


Define the inner triangle and the 3 corner triangles

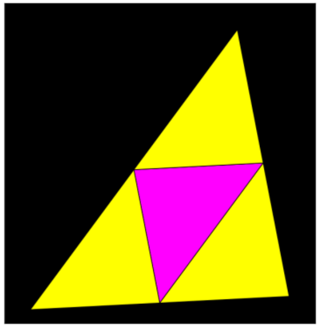


Color the inner triangle and repeat the process on the 3 corner triangles

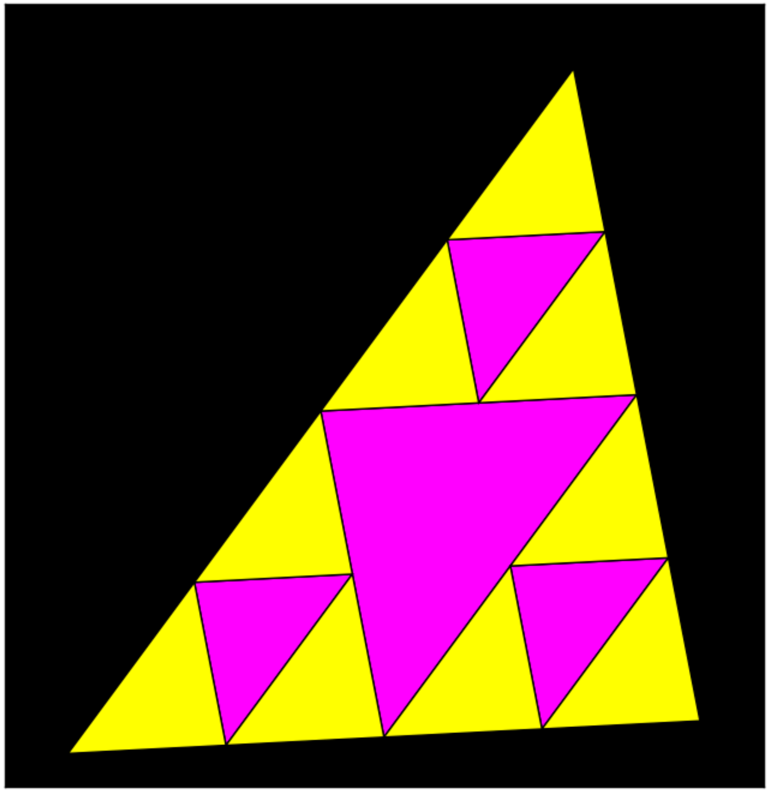
“Repeat the Process”



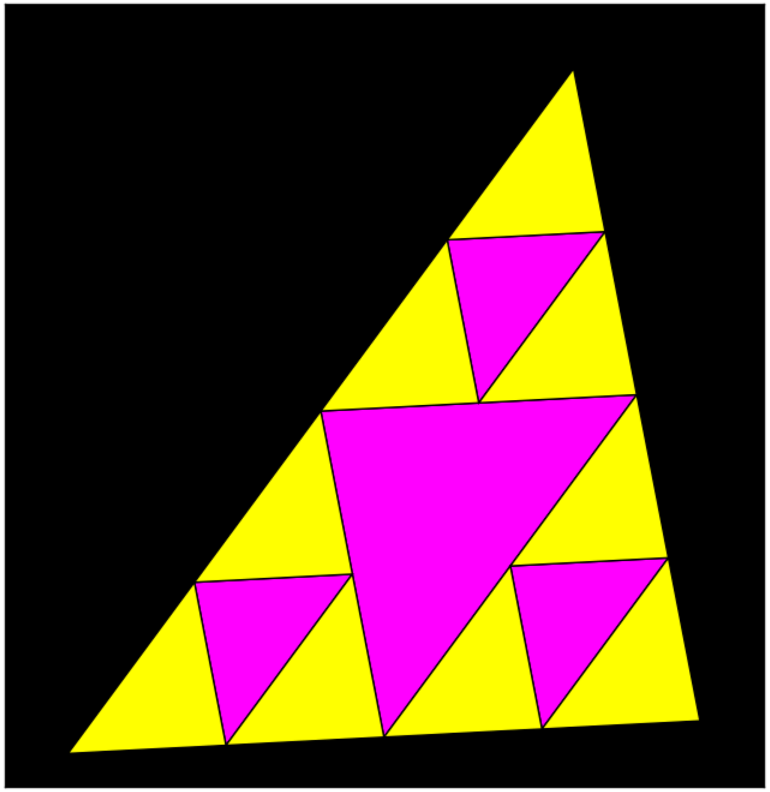
Visit every yellow triangle and replace it with this



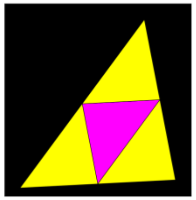
We Get This...



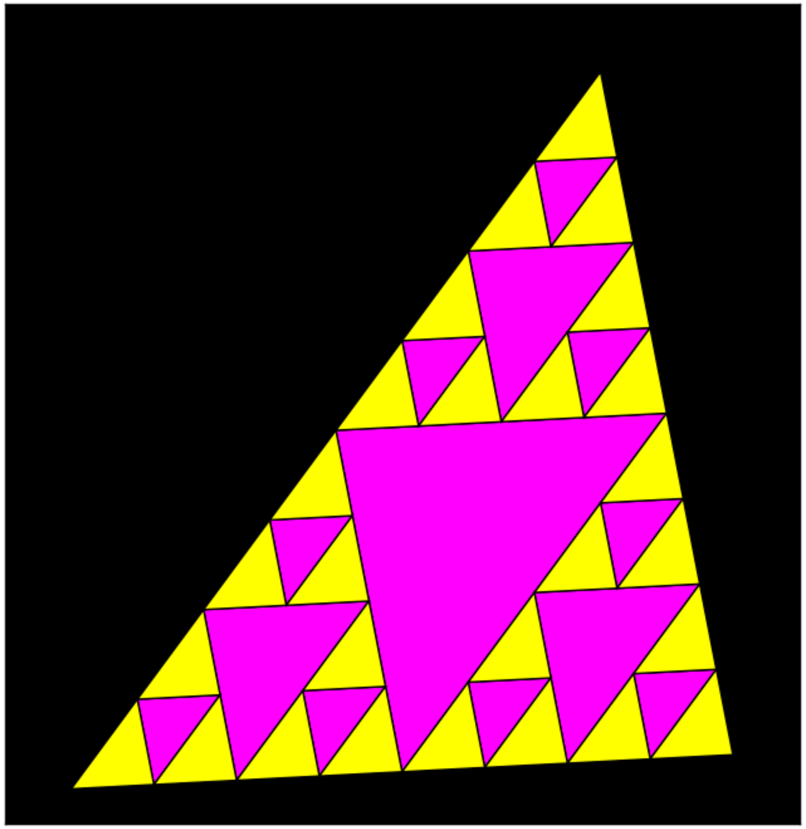
"Repeat the Process"



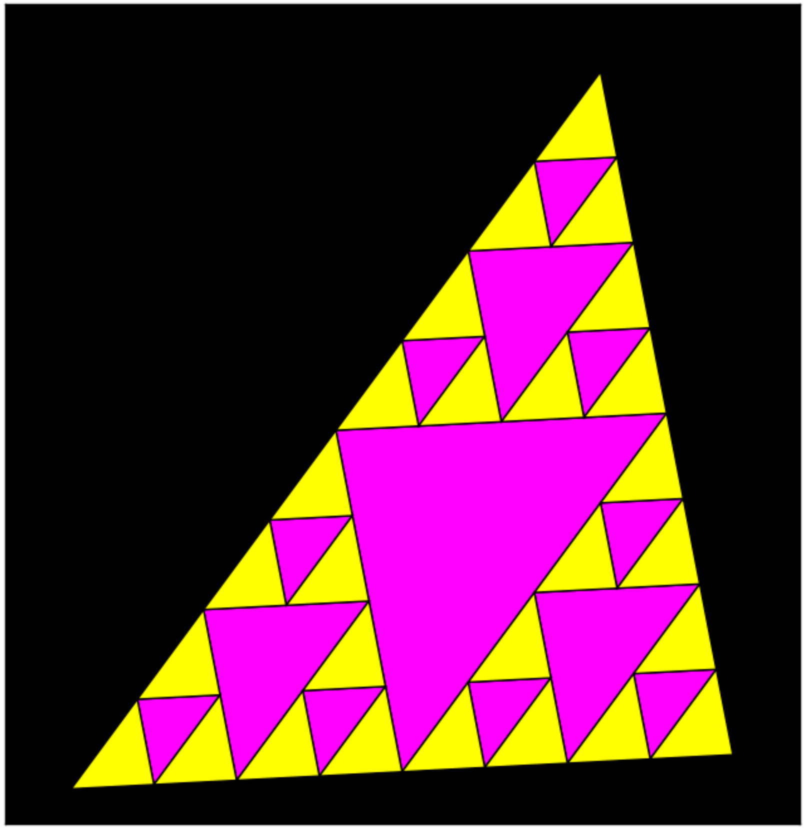
Visit every yellow triangle and replace it with



We Get This…

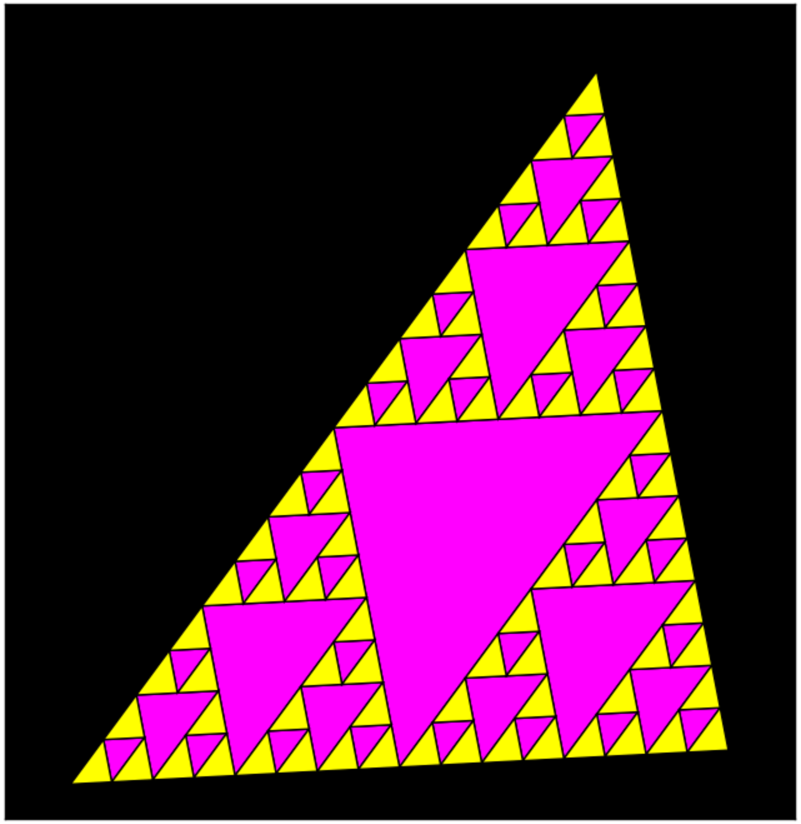


"Repeat the Process"



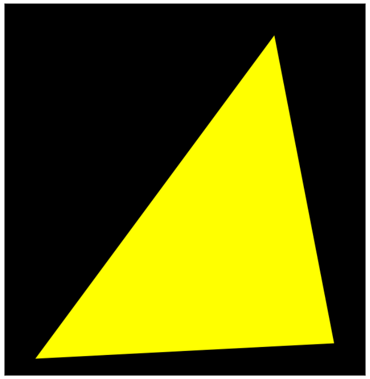
Visit every yellow triangle and replace it with

We Get This…

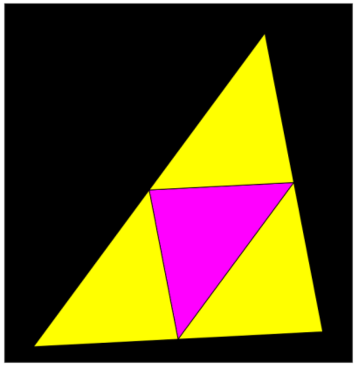


Etc.

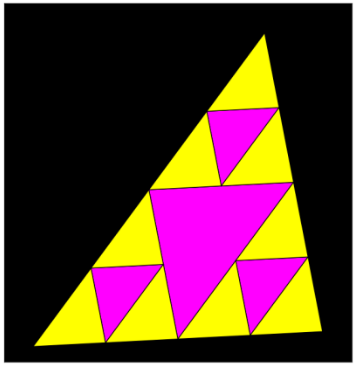
The Notion of Level



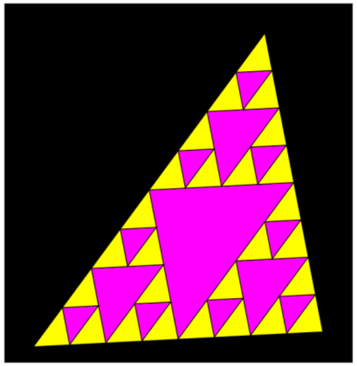
A 0-level tiling



A 1-level tiling



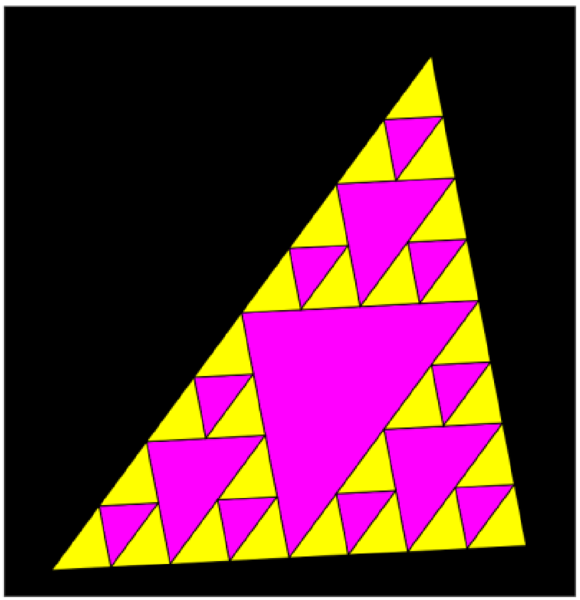
A 2-level tiling

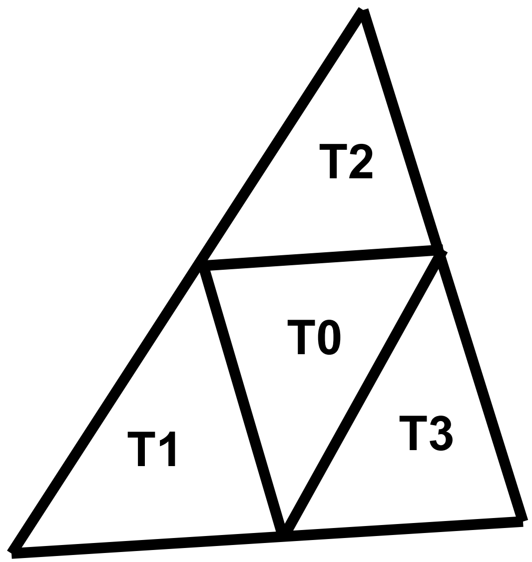


A 3-level tiling

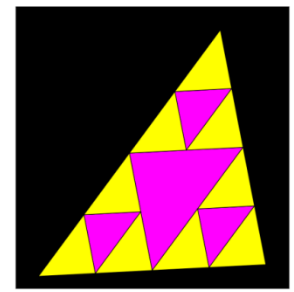
The Connection Between Levels

A **3**-level tiling





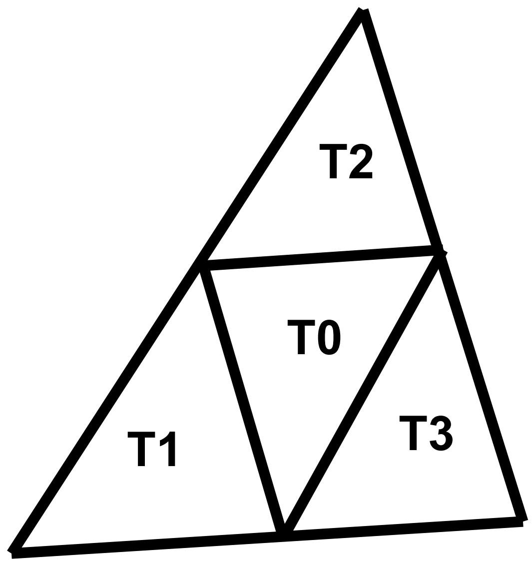
A **2**-level tiling



To display a **3**-level tiling you do this:

- display the inner triangle T0
- display a **2**-level tiling of corner triangles T1, T2, and T3

The Connection Between Levels



To display an **N**-level tiling you do this:
- display the inner triangle T0
- display an **(N-1)-**level tiling of triangles T1, T2, and T3

A Recursive Procedure

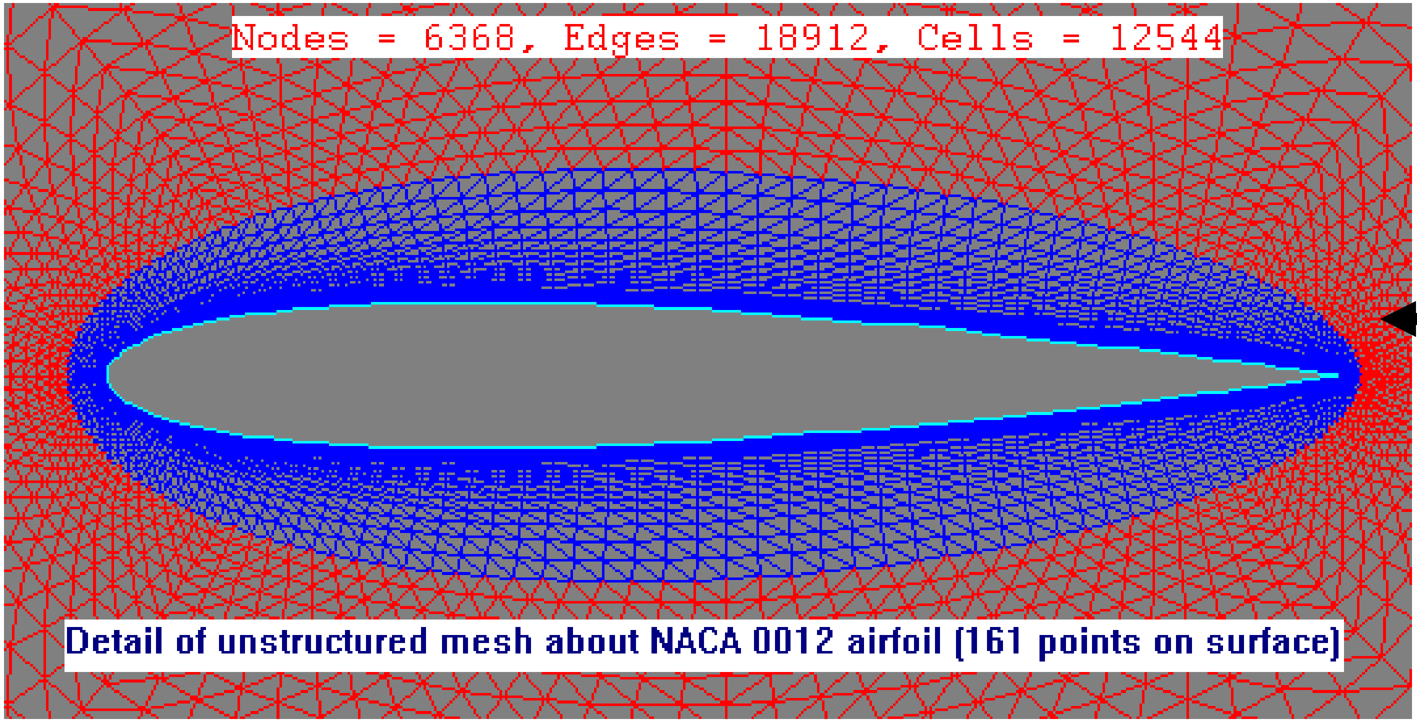
```
def Tile(T,level):
    # PreC: T a triangle
    if Level ==0:
      Draw T (yellow)
    else:
      # Let T0 be the inner triangle and
      # T1,T2,and T3 be the corner triangles
      Draw T0 (magenta)
      Tile(T1,level-1)
      Tile(T2,level-1)
      Tile(T3,level-1)
```

This is the “base case”.
A 0-level tiling just draws the input triangle

These are the recursive procedure calls.
The procedure Tile calls itself three times.

A Note on Chopping up a Region into Triangles…

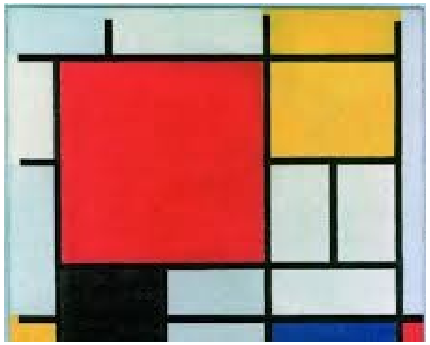
It is Important!

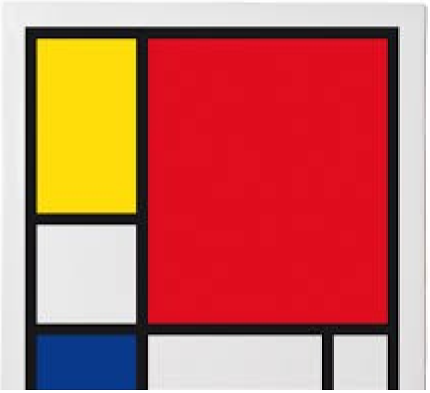


An
Area
Of
Interest

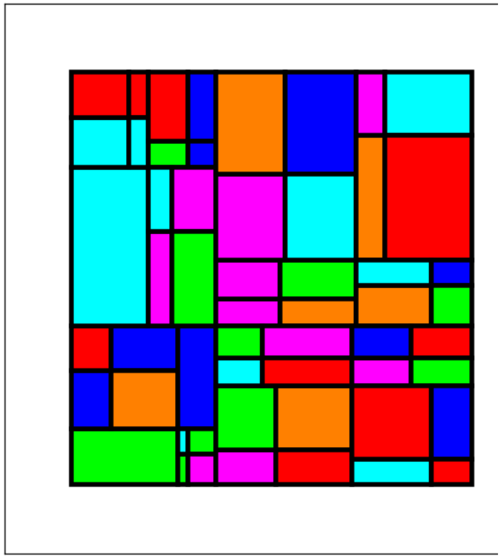
Step One in simulating flow around an airfoil is to generate a triangular mesh and (say) estimate the velocity at each little triangle using physics and math.

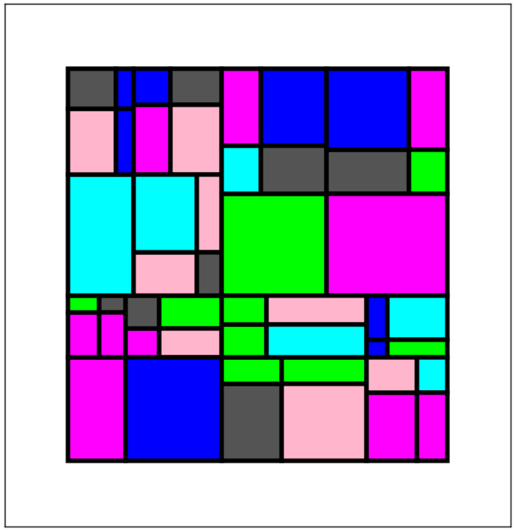
Another Example: Random Mondrians





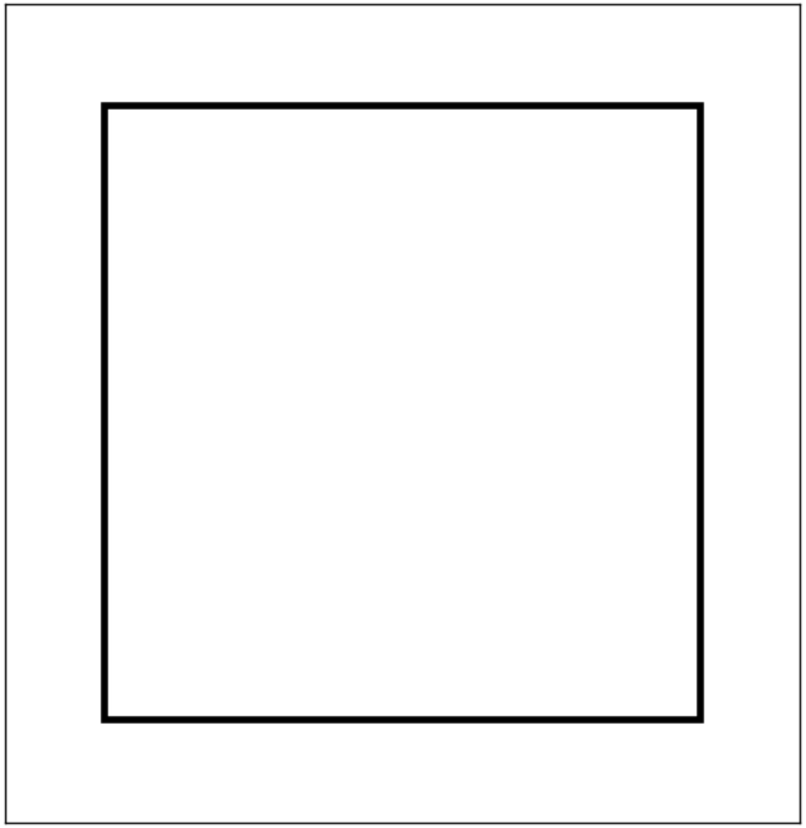
Using Python:





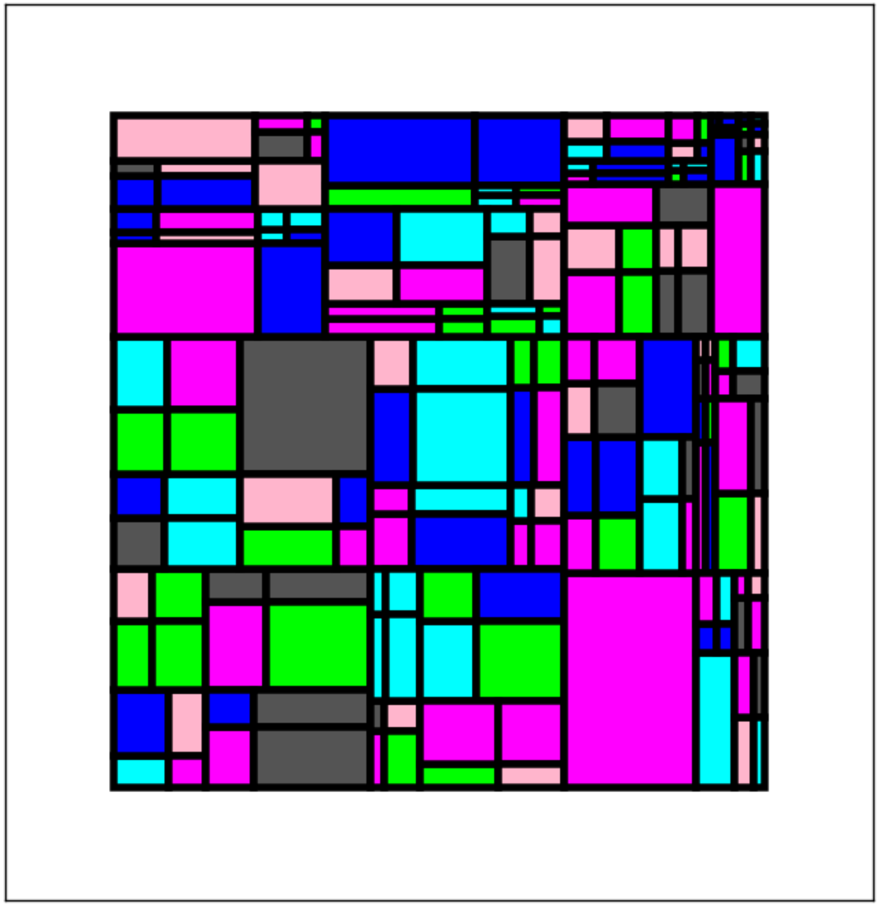
Random Mondrian

Given This:

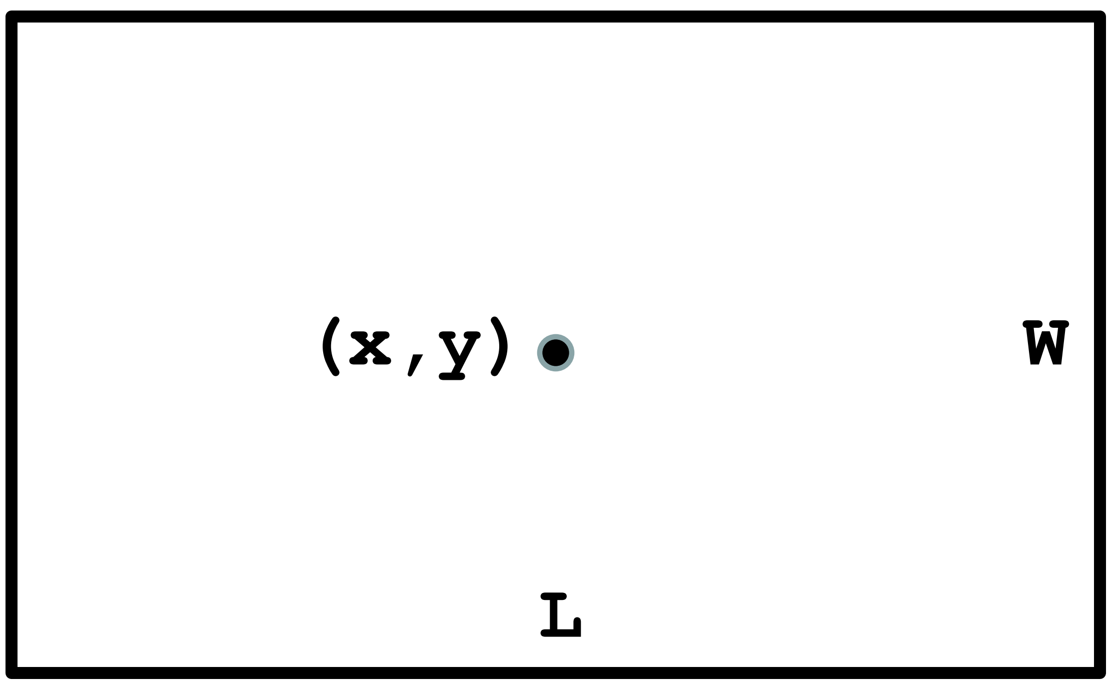


Random Mondrian

Draw This:

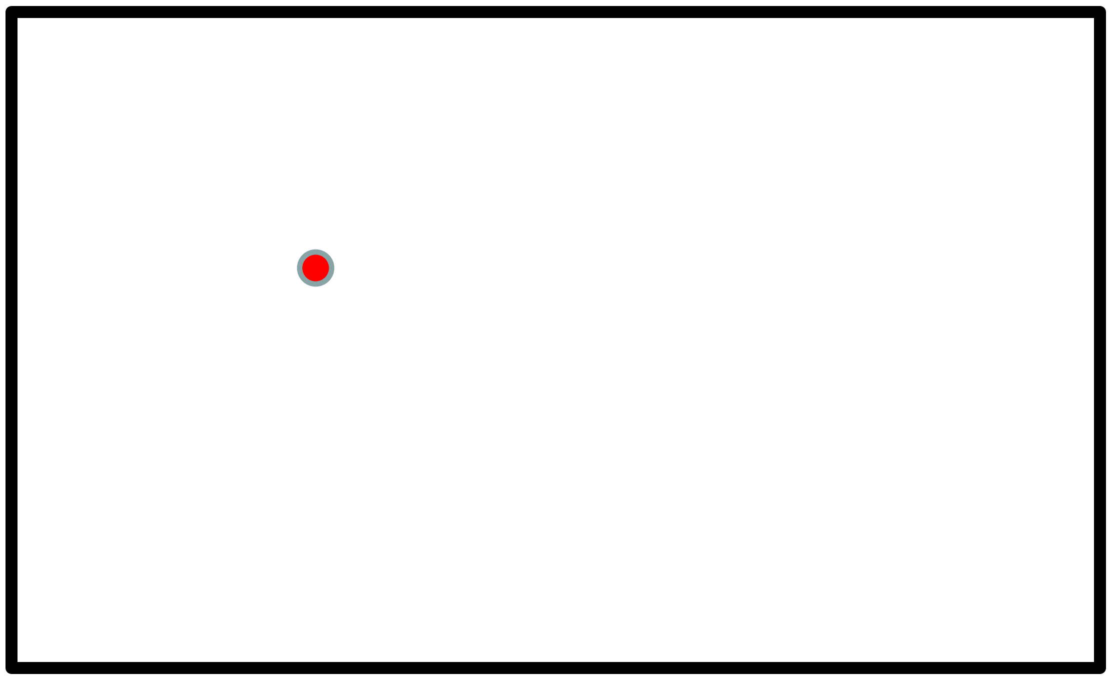


The Subdivide Process Applies to a Rectangle

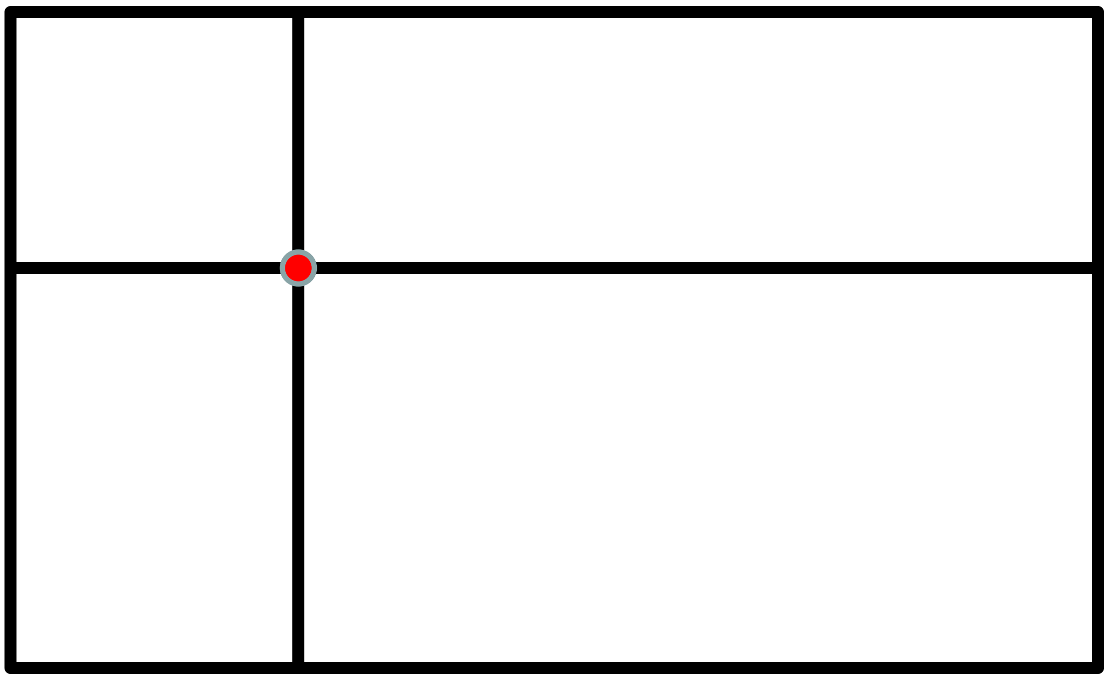


Given a rectangle specified by its length, width, and center, either randomly color it or randomly subdivide it.

Subdivision Starts with a Random Dart Throw

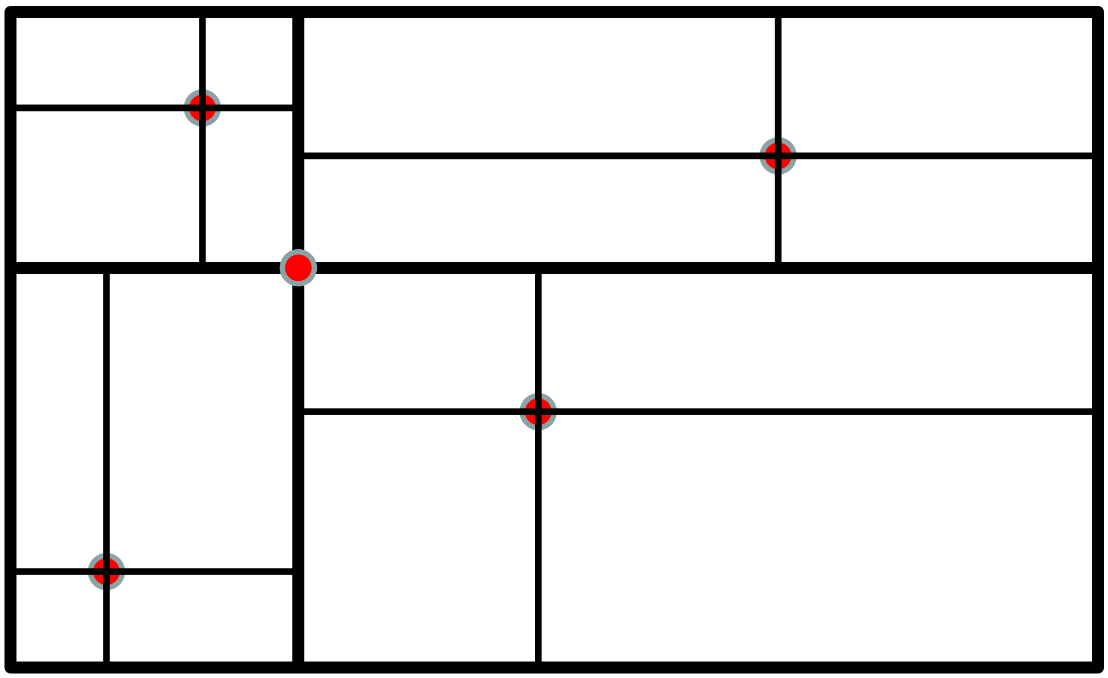


This Defines 4 Smaller Rectangles



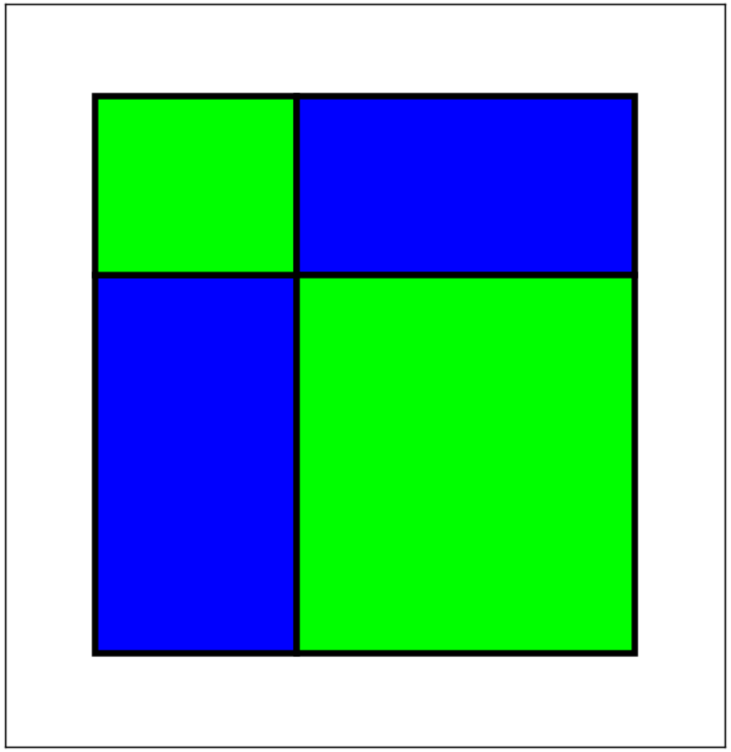
Repeat the process on each of the 4 smaller rectangles...

This Defines 4 Smaller Rectangles

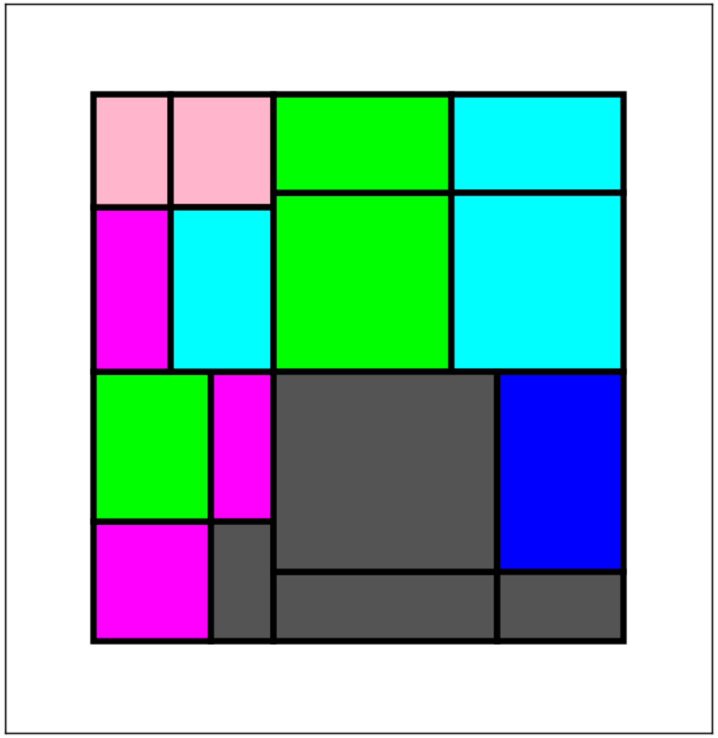


We can again repeat the process on each of the 16 smaller rectangles. Etc.

The Notion of Level



A 1-level Partitioning



A 2-level Partitioning

Pseudocode

```
def Mondrian(x,y,L,W,level):
    if level ==0:
        c = RandomColor())
        DrawRect(x,y,L,W,FillColor=c)
    else:
      # Subdivide into 4 smaller rectangles
        Mondrian(upper left rectangle info,level-1)
        Mondrian(upper right rectangle info,level-1)
        Mondrian(lower left rectangle info,level-1)
        Mondrian(lower right rectangle info,level-1)
```

We look at a few details. Complete implementation online

How to Generate Random Colors

We need some new technology to organize the selection random colors.

We need lists whose entries are lists.

Lists with Entries that Are Lists

An Example:

```
cyan      = [0.0,1.0,1.0]
magenta   = [1.0,0.0,1.0]
yellow    = [1.0,1.0,0.0]
colorList = [cyan,magenta,yellow]
```

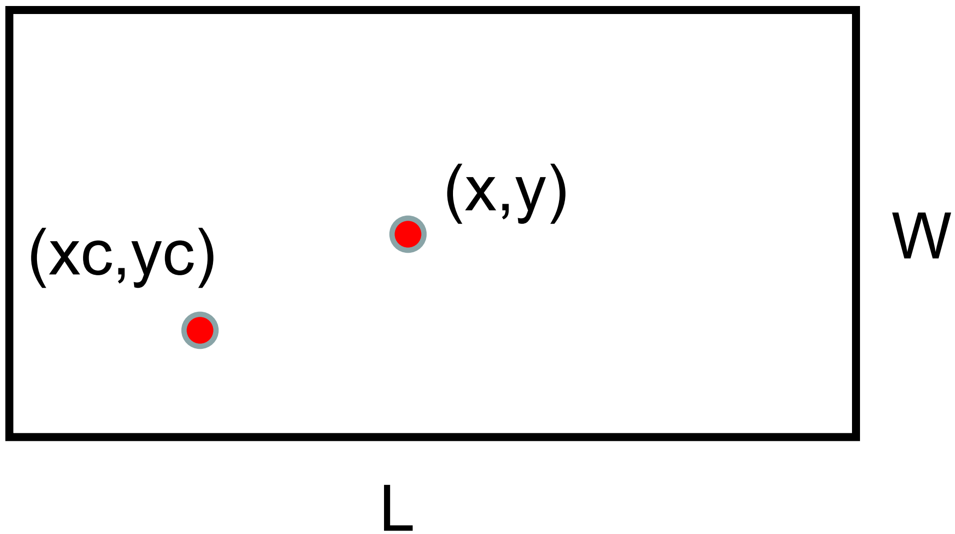
Pick a Color at Random

```
cyan      = [0.0,1.0,1.0]
magenta   = [1.0,0.0,1.0]
yellow    = [1.0,1.0,0.0]
colorList = [cyan,magenta,yellow]
r = randi(0,2)
randomColor = colorList[r]
```

Package the Idea…

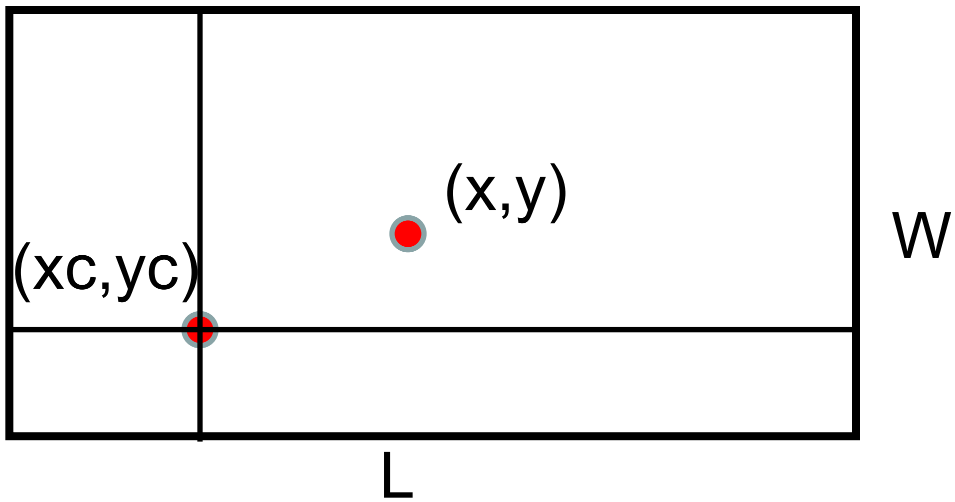
```
from simpleGraphics import *
from random import randint as randi

def RandomColor():
    """ Returns a randomly selected
    rgb list."""
    c = [RED,GREEN,BLUE,ORANGE,CYAN]
    i = randi(0,len(c)-1)
    return c[i]
```

How to Randomly Subdivide a Rectangle




```
xc = randu(x-L/2,x+L/2)
yc = randu(y-W/2,y+W/2)
```

The Math Behind the Little Rectangles



The upper right rectangle is typical:

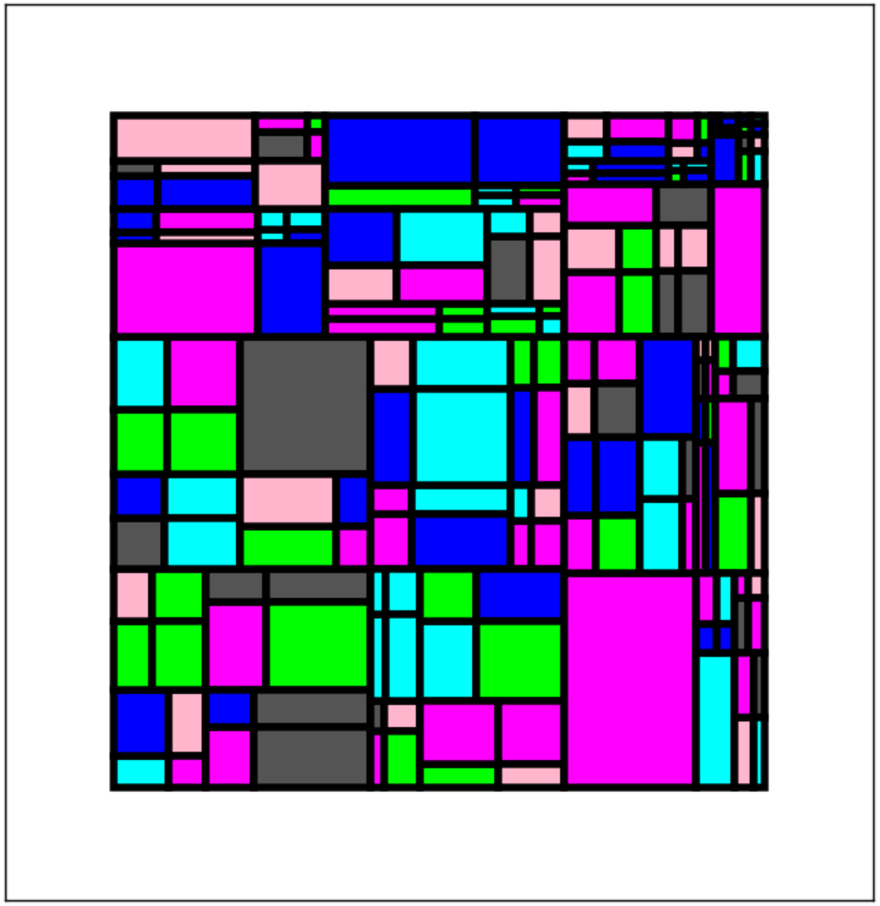
Length:	`L1 = (x+L/2)-xc`
Width:	`W1 = (y+W/2)-yc`
Center:	`(xc+L1/2,yc+W1/2)`

The Procedure **Mondrian**

A couple of features
to make the design more
interesting:

(1) The dart throw that
determines the subdivision
can't land too near the edge.
No super skinny tiles!

(2) Randomly decide
whether or not to subdivide.
This creates a nice diversity
in size.



Next Up

A Non-Graphics Example of Recursion: The Factorial Function

Recursive Evaluation of Factorial

Recall the factorial function:

```
def F(n):
    x = 1
    for k in range(1,n+1):
        x = x*k
    return x
```

5! = 1x2x3x4x5

Recursive Evaluation of Factorial

Q. How would you compute 6! given that you have computed 5! = 120 ?

A. 6! = 120 x 6

5! = 1x2x3x4x5

Recursive Evaluation of Factorial

```
def F(n):
    if n<=1:
        return 1
    else:
        a = F(n-1)
        return n*a
```

How does this work?

Executing F(3)

```
m = 3
x = F(m)
print x
```

```
m --> 3

x --> 
```

We are in the calling script

Executing F(3)

```
m = 3
x = F(m)
print
```

```
def F(n):
   if n<=1:
      return 1
   else:
      a = F(n-1)
      return n*a
```

```
m --> 3

x -->
```

```
n -->    3

a -->

return
```

The function F is called with argument 3. We open up a call frame.

Executing F(3)

```
m = 3
x = F(m)
print x
```

```
def F(n):
   if n<=1:
      return 1
   else:
      a = F(n-1)
      return n*a
```

```
m -->  3
x -->
```

```
n -->   3
a -->
return
```

We encounter a function call. F is called with argument equal to 2.

Executing F(3)

```
m = 3
x = F(m)
print x
```

```
def F(n):
   if n<=1:
      return 1
   else:
      a = F(n-1)
      return n*a
```

```
def F(n):
   if n<=1:
      return 1
   else:
      a = F(n-1)
      return n*a
```

m --> 3
x -->

n --> 3
a -->
return

n --> 2
a -->
return

We open up a call frame.

Executing F(3)

```
m = 3
x = F(m)
print x
```

```
def F(n):
   if n<=1:
      return 1
   else:
      a = F(n-1)
      return n*a
```

```
def F(n):
   if n<=1:
      return 1
   else:
      a = F(n-1)
      return n*a
```

m -->	3
x -->	

n -->	3
a -->	
return	

n -->	2
a -->	
return	

We encounter a function call. F is called with argument 1

Executing F(3)

```
m = 3
x = F(m)
print x
```

```
def F(n):
   if n<=1:
      return 1
   else:
      a = F(n-1)
      return n*a
```

```
def F(n):
   if n<=1:
      return 1
   else:
      a = F(n-1)
      return n*a
```

```
def F(n):
   if n<=1:
      return 1
   else:
      a = F(n-1)
      return n*a
```

```
m --> 3
x -->
```

```
n --> 3
a -->
return
```

```
n --> 2
a -->
return
```

```
n --> 1
a -->
return
```

We open up a call frame.

Executing F(3)

```
m = 3
x = F(m)
print x
```

```
def F(n):
   if n<=1:
      return 1
   else:
      a = F(n-1)
      return n*a
```

```
def F(n):
   if n<=1:
      return 1
   else:
      a = F(n-1)
      return n*a
```

```
def F(n):
   if n<=1:
      return 1
   else:
      a = F(n-1)
      return n*a
```

```
m --> 3
x -->
```

```
n -->  3
a -->
return
```

```
n -->  2
a -->
return
```

```
n -->  1
a -->
return 1
```

The value of 1 is "assigned" to return

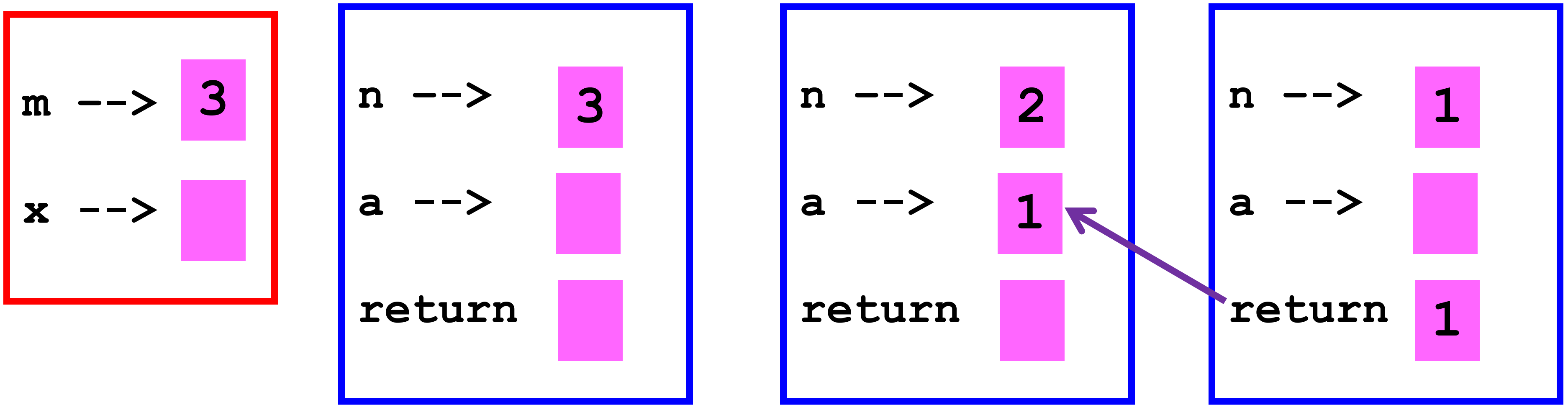
Executing F(3)

```
m = 3
x = F(m)
print x
```

```
def F(n):
   if n<=1:
      return 1
   else:
      a = F(n-1)
      return n*a
```

```
def F(n):
   if n<=1:
      return 1
   else:
      a = F(n-1)
      return n*a
```

```
def F(n):
   if n<=1:
      return 1
   else:
      a = F(n-1)
      return n*a
```



The value is sent back to the caller.

Executing F(3)

```
m = 3
x = F(m)
print x
```

```
def F(n):
   if n<=1:
      return 1
   else:
      a = F(n-1)
      return n*a
```

```
def F(n):
   if n<=1:
      return 1
   else:
      a = F(n-1)
      return n*a
```

```
def F(n):
   if n<=1:
      return 1
   else:
      a = F(n-1)
      return n*a
```

m --> 3

x -->

n --> 3

a -->

return

n --> 2

a --> 1

return

n --> 1

a -->

return 1

That function call is over

Executing F(3)

```
m = 3
x = F(m)
print x
```

```
def F(n):
   if n<=1:
      return 1
   else:
      a = F(n-1)
      return n*a
```

```
def F(n):
   if n<=1:
      return 1
   else:
      a = F(n-1)
      return n*a
```

```
m --> 3
x -->
```

```
n --> 3
a -->
return
```

```
n --> 2
a --> 1
return
```

Control now passes to this “edition” of F

Executing F(3)

```
m = 3
x = F(m)
print x
```

```
def F(n):
   if n<=1:
      return 1
   else:
      a = F(n-1)
      return n*a
```

```
def F(n):
   if n<=1:
      return 1
   else:
      a = F(n-1)
      return n*a
```

```
m --> 3
x -->
```

```
n -->      3
a -->
return
```

```
n -->      2
a -->      1
return     2
```

Control passes to this “edition” of F. The value 2 is “assigned” to return

Executing F(3)

```
m = 3
x = F(m)
print x
```

```
def F(n):
   if n<=1:
      return 1
   else:
      a = F(n-1)
      return n*a
```

```
def F(n):
   if n<=1:
      return 1
   else:
      a = F(n-1)
      return n*a
```

```
m --> 3
x -->
```

```
n --> 3
a --> 2
return
```

```
n --> 2
a --> 1
return 2
```

The value is returned to the caller.

Executing F(3)

```
m = 3
x = F(m)
print x
```

```
def F(n):
   if n<=1:
      return 1
   else:
      a = F(n-1)
      return n*a
```

```
def F(n):
   if n<=1:
      return 1
   else:
      a = F(n-1)
      return n*a
```

```
m -->  3
x -->
```

```
n -->  3
a -->  2
return
```

```
n -->  2
a -->  1
return 2
```

The function call is over

Executing F(3)

```
m = 3
x = F(m)
print x
```

```
def F(n):
   if n<=1:
      return 1
   else:
      a = F(n-1)
      return n*a
```

m --> 3

x -->

n --> 3

a --> 2

return

Control now passes to this “edition” of F

Executing F(3)

```
m = 3
x = F(m)
print x
```

```
def F(n):
   if n<=1:
      return 1
   else:
      a = F(n-1)
      return n*a
```

```
m -->  3

x -->
```

```
n -->      3

a -->      2

return     6
```

The value 6 is "assigned" to return

Executing F(3)

```
m = 3
x = F(m)
print x
```

```
def F(n):
   if n<=1:
      return 1
   else:
      a = F(n-1)
      return n*a
```

```
m --> 3
x --> 6
```

```
n --> 3
a --> 2
return 6
```

The value is returned to the caller.

Executing F(3)

```
m = 3
x = F(m)
print x
```

```
def F(n):
   if n<=1:
      return 1
   else:
      a = F(n-1)
      return n*a
```

```
m -->  3
x -->  6
```

```
n -->     3
a -->     2
return    6
```

This function call is over.

Executing F(3)

```
m = 3
x = F(m)
print x
```

```
m --> 3

x --> 6
```

Control passes to the script that asked for F(3)

Executing F(3)

```
m = 3
x = F(m)
print x
```

```
m --> 3

x --> 6
```

Output: 6

All Done!

Overall Conclusions

Recursion is sometimes the simplest way to organize a computation.

It would be next to impossible to do the triangle tiling problem any other way.

On the other hand, factorial computation is easier via for-loop iteration.

Overall Conclusions

Infinite recursion (like infinite loops) can happen so careful reasoning is required.

Will we reach the “base case”?

Graphics examples: We will reach Level==0

Factorial: We will reach n==1