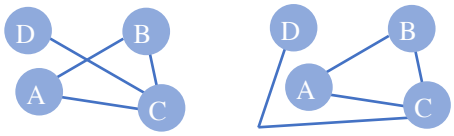
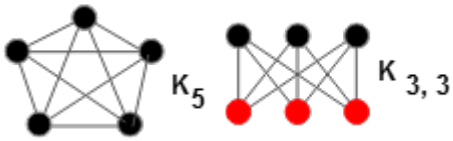
Planar Graphs

A graph is *planar* if it can be drawn in the plane without any edges crossing.



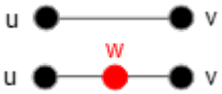
Be careful with this definition! The first graph on the right doesn't look planar because edges (D, C) and (A, B) cross. But edge (D, C) can be redrawn as shown in the second version of the same graph on the right, so this graph *is* planar.

In 1930, Polish mathematician Kazimierz Kuratowski published this theorem:



> A finite graph is planar if and only if it does not contain a subgraph that is a "subdivision" of K_5 (the complete graph of five vertices) or of $K_{3,3}$ (complete bipartite graph of 6 vertices).

According to Wikipedia, Orrin Frink and Paul Smith independently proved the theorem in 1930, as did the Russian Lev Pontryagin in 1927, but their proofs were not published, at least immediately. Thus, the proof is generally called *Kuratowski's theorem*.



We have to explain what "subdivision" means. Consider edge {u, v} to the right. Split the edge into two edges by inserting a new node w, as shown below edge {u, v}. We have created a subdivision of edge {u, v}. A subdivision of a graph can contain many subdivisions of its edges. Further, the newly introduced nodes can contain other edges leaving them.

The opposite of subdividing is called *smoothing*. You can imagine that it is not easy to smooth a graph to end up seeing that it contains K_5 or $K_{3,3}$, so other ways must be found to see whether a graph is planar.

The first linear-time planarity algorithm

In 1970-71, John Hopcroft of the Cornell Computer Science Department spent a year at Stanford. Polya Hall, where Stanford's CS Department lived, was crowded, and he had to share a room with a PhD student, Bob Tarjan. This was actually quite fortuitous! They were both interested in algorithms and data structures. Together, they worked on an algorithm to find out whether a graph was planar and to produce a planar embedding if it was. Thus was born the first linear (in the number of edges) algorithm for testing planarity.

For this and many more contributions, John and Bob were awarded the 1986 ACM Turing Award. The citation reads: "For fundamental achievements in the design and analysis of algorithms and data structures."

Their work was done before we knew much about how to present algorithms, and this is a very difficult algorithm to understand. Thus, their paper *Efficient planarity testing* (J ACM 21, 4 (Oct 1974), 549–568) is not for the faint hearted. In 1988, David Gries and Jinyun Xue wrote CS Technical Report 88-906, *The Hopcroft-Tarjan Planarity Algorithm, Presentations and Improvements*. Look at JavaHyperText entry "planarity" for a copy.

But OO programming was not yet popular in 1986, and Java did not yet exist! Gries and Xue used Pascal. A good exercise would be to read the Gries-Xue presentation and rewrite it in Java.

Search the web for "planarity testing" and you can find more on the subject.