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## 1 Sylvester equations

The *Sylvester equation* (or the special case of the *Lyapunov equation*) is a matrix equation of the form

$$AX + XB = C$$

where $A \in \mathbb{R}^{m\times m}, B \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{m\times n}$, are known, and $X \in \mathbb{R}^{m\times n}$ is to be determined. The Sylvester equation has important applications in control theory, and also plays a prominent role in the theory of several classes of structured matrices.

On the surface of it, this is a simple system: the expressions $AX$ and $XB$ are just linear in the elements of $X$, after all. Indeed, we can rewrite the system as

$$(I \otimes A + B^T \otimes I) \operatorname{vec}(X) = \operatorname{vec}(C),$$

where

$$\operatorname{vec}(\begin{bmatrix} x_1 & \dots & x_n \end{bmatrix}) = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

is a vector of length $mn$ composed by listing the elements of $X$ in column-major order, and the *Kronecker product* is defined by

$$F \otimes G = \begin{bmatrix} f_{11}G & f_{12}G & \dots \\ f_{21}G & f_{22}G & \dots \\ \vdots & \vdots & \ddots \end{bmatrix}.$$

Alas, solving this matrix equation by Gaussian elimination would cost $O((mn)^3)$. Can we do better?

The *Bartels-Stewart* algorithm is a clever approach to the problem that takes only $O(\max(m, n)^3)$ time. The key is to compute the Schur factorizations

$$A = U_A T_A U_A^* \qquad B = U_B T_B U_B^*$$

from which we obtain

$$T_A \tilde{X} + \tilde{X} T_B = \tilde{C}$$

where $\tilde{X} = U_A^* X U_B$ and $\tilde{C} = U_A^* C U_B$. Column $j$ of this system of equations can be written as

$$(T_A + t_{B,kk} I)\tilde{x}_k = \tilde{c}_k - \sum_{j=1}^{k-1} \tilde{x}_j t_{B,jk};$$

therefore, we can solve each column of $\tilde{x}$ in turn by a back-substitution procedure that involves a triangular linear solve. We only run into trouble if one of these systems is singular (or nearly so), corresponding to the case where $A$ and $-B$ (nearly) have an eigenvalue in common.

## 1.1 Riccati equations

The Sylvester equation is a *linear* matrix equation whose solution is accelerated via an intermediate eigendecomposition. The *algebraic Riccati equation* is a *quadratic* matrix equation that also can be expressed via an eigenvalue problem. The Riccati equation occurs in optimal control problems, as well as some other places; for the continuous-time optimal control problem, we would usually write

$$A^T X + XA - XBR^{-1}B^T X + Q = 0$$

where $R$ and $Q$ are spd matrices representing cost functions, $A$ and $B$ are general square matrices, and we seek a symmetric solution matrix $X$.

The key to thinking of the Riccati equation via eigenvalues is to write the left hand side of the equation as a pure quadratic:

$$\begin{bmatrix} I \\ X \end{bmatrix} \begin{bmatrix} Q & A \\ A^T & -BR^{-1}B^T \end{bmatrix} \begin{bmatrix} I \\ X \end{bmatrix} = 0.$$

We can also characterize this by the relation

$$\begin{bmatrix} Q & A \\ A^T & -BR^{-1}B^T \end{bmatrix} \begin{bmatrix} I \\ X \end{bmatrix} = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \begin{bmatrix} I \\ X \end{bmatrix},$$

or, equivalently

$$Z = \begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix}, \quad Z \begin{bmatrix} I \\ X \end{bmatrix} = \begin{bmatrix} I \\ X \end{bmatrix} L.$$

That is, we want a specific basis of an invariant subspace of a *Hamiltonian matrix*, i.e. a matrix $Z$ such that

$$JZ \text{ symmetric}, J \equiv \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}.$$

Hamiltonian eigenvalue problems show up in a surprising variety of places in addition to optimal control. The theory of eigenvalue problems for Hamiltonian and skew-Hamiltonian matrices is reasonably well developed, and the eigenvalues have a special symmetry to them. There is now good software for these classes of problems that exploits the structure — though not in LAPACK. The right place to look for these solvers is in the SLICOT package.

# 2 Polynomial eigenvalue problems

A *nonlinear eigenvalue problem* is an equation of the form

$$T(\lambda)v = 0$$

where $T : \mathbb{C} \to \mathbb{C}^{n\times n}$ is a matrix-valued function. The most common nonlinear eigenvalue problems are *polynomial eigenvalue problems* in which $T$ is a polynomial; and most common among the polynomial eigenvalue problems are the *quadratic eigenvalue problems*

$$(\lambda^2 M + \lambda D + K)u = 0.$$

As the notation might suggest, one of the natural sources of quadratic eigenvalue problems is in the analysis of damped unforced vibrations in mechanical (or other physical) systems. In this context, $M$, $D$, and $K$ are the *mass*, *damping*, and *stiffness* matrices, and the eigenvalue problem arises from the search for special solutions to the equation

$$M\ddot{x} + D\dot{x} + Kx = 0$$

where $x(t) = u\exp(\lambda t)$. We note that the mass matrix is often symmetric and positive definite; in this case, we can apply a change of variables to convert to a problem in which the leading term involves an identity matrix. We will assume this case for the remainder of our discussion.

When studying the solution of higher-order differential equations, a standard trick is to put the system into first-order form by introducing auxiliary variables for derivatives. For example, we would put our model second-order unforced vibration equation into first-order form by introducing the variable $v = \dot{x}$; then (assuming $M = I$), we have

$$\begin{bmatrix} \dot{v} \\ \dot{x} \end{bmatrix} = \begin{bmatrix} -D & -K \\ I & 0 \end{bmatrix} \begin{bmatrix} v \\ x \end{bmatrix}.$$

Similarly, we can convert the quadratic eigenvalue problem into a standard linear eigenvalue problem by introducing $w = \lambda u$; then

$$\lambda \begin{bmatrix} w \\ u \end{bmatrix} = \begin{bmatrix} -D & -K \\ I & 0 \end{bmatrix} \begin{bmatrix} w \\ u \end{bmatrix}.$$

This process of converting a quadratic (or higher-order polynomial) eigenvalue problem into a linear eigenvalue problem in a higher-dimensional space is called *linearization* (a somewhat unfortunate term, but the standard choice). There are many ways to define the auxiliary variables, and hence many ways to linearize a polynomial eigenvalue problem; the version we have described is the *companion* linearization. Different linearizations are appropriate to polynomial eigenvalue problems with different structure.

More generally, a "genuinely" nonlinear eigenvalue involves a matrix $T(\lambda)$ that depends on the spectral parameter $\lambda$ as a more general non-rational function. Typically, we restrict our attention to functions that are complex-analytic in some domain of interest; these arise naturally in many applications, particularly in problems involving delay, radiation, and similar effects. One thread in my own research has been to extend some of the theory we have for the standard eigenvalue problem — results like Gershgorin and Bauer-Fike — to this more general nonlinear case.

# 3 Pseudospectra

We conclude our discussion of eigenvalue-related ideas by revisiting the perturbation theory for the nonsymmetric eigenvalue problem from a somewhat different perspective. In the symmetric case, if $A - \hat{\lambda} I$ is nearly singular (i.e. $(A - \hat{\lambda} I)\hat{x} = r$ where $\|r\| \ll \|A\| \|\hat{x}\|$), then $\hat{\lambda}$ is close to one of the eigenvalues of $A$. But in the nonsymmetric case, $A - \hat{\lambda} I$ may become quite close to singular even though $\hat{\lambda}$ is quite far from any eigenvalues of $A$. The approximate null vector of $A - \hat{\lambda} I$ is sometimes called a *quasi-mode*, and dynamical systems defined via such a matrix $A$ are often characterized by long-lived transient dynamics that are well-described in terms of such quasi-modes.

In order to describe quasi-modes and long-lived transients, we need a systematic way of thinking about "almost eigenvalues." This leads us to the idea of the $\epsilon$-*pseudospectrum*:

$$\Lambda_\epsilon(A) = \{z \in \mathbb{C} : \|(A - zI)^{-1}\| \geq \epsilon^{-1}\}.$$

This is equivalent to

$$\Lambda_\epsilon(A) = \{z \in \mathbb{C} : \exists E \text{ s.t. } \|E\| < \epsilon \text{ and } (A + E - zI) \text{ singular}\},$$

or, when the norm involved is the operator 2-norm,

$$\Lambda_\epsilon(A) = \{x \in \mathbb{C} : \sigma_{\min}(A - zI) < \epsilon\}.$$

There is a great deal of beautiful theory involving pseudospectra; as a guide to the area, I highly recommend *Spectra and Pseudospectra* by Mark Embree and Nick Trefethen.