HW for 2020-01-31

(due: 2020-02-07)

You may (and should) talk about problems with each other and with me, providing attribution for any good ideas you might get. Your final write-up should be your own.

1: Conditioning For a differentiable function $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$, we define the (2-norm) condition number at x to be

$$\kappa_2(x) = \frac{\|J(x)\|_2 \|x\|_2}{\|f(x)\|_2}$$

where $J(x)$ is the *Jacobian* matrix (the matrix of partial derivatives). This generalizes the scalar condition number defined in the notes. Given this definition, what is the condition number of $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ given by $f(x) = x_1 x_2$? Under what circumstances is this condition number large?

2: Box-Cox The Box-Cox family of transformations is sometimes used in statistical applications to normalize non-negative data. The transformation has the form

$$g_\lambda(x) = \begin{cases} \frac{x^\lambda - 1}{\lambda}, & \lambda \neq 0 \\ \log x, & \lambda = 0 \end{cases}$$

The most obvious implementation (in MATLAB) is

```
if lambda == 0
  g = log(x);
else
  g = (x^lambda-1)/lambda;
end
```

Unfortunately, this is prone to large relative errors when $|\lambda \log x| \ll 1$. Explain why, and suggest an alternative with better error in this case. You may wish to use the function expm1 for your solution (though there are other approaches as well).

3: Pi, see! The following routine estimates π by recursively computing the semiperimeter of a sequence of 2^{k+1}-gons embedded in the unit circle:

```
N = 4;
L(1) = sqrt(2);
s(1) = N*L(1)/2;
for k = 1:30
  N = N*2;
  L(k+1) = sqrt( 2*(1-sqrt(1-L(k)^2/4)) );
  s(k+1) = N*L(k+1)/2;
end

semilogy(1:length(s), abs(s-pi));
ylabel('|s_k-\pi|');
xlabel('k')
```

Plot the absolute error $|s_k - \pi|$ against k on a semilog plot. Explain why the algorithm behaves as it does, and describe a reformulation of the algorithm that does not suffer from this problem.