Principled Programming

Introduction to Coding in Any Imperative Language

Tim Teitelbaum

Emeritus Professor

Department of Computer Science

Cornell University

Median

	0	1	2	3	4	n
A	10	20	**30**	40	50	

	0	1	2	3	4	5	**n**
A	10	20	30	**40**	50	60	

The *median* of an ordered array of n values is the middle value. If n is odd, this is A[n/2]; if n is even, we also opt for A[n/2] rather than averaging the middle two values.

	0	1	2	3	4	n
A	50	**30**	10	40	20	

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The *median* of an ordered array of n values is the middle value. If n is odd, this is $A[n/2]$; if n is even, we also opt for $A[n/2]$ rather than averaging the middle two values.

But what if the array is not ordered. How would you find the median then?

You could sort the array and select $A[n/2]$. But sorting requires n log n operations.

Is it possible to do better? Try it. You will find that everyday experience is no help.

	0	1	2	3	4	n
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The *median* of an ordered array of *n* values is the middle value. If *n* is odd, this is $A[n/2]$; if *n* is even, we also opt for $A[n/2]$ rather than averaging the middle two values.

But what if the array is not ordered. How would you find the median then?

You could sort the array and select $A[n/2]$. But sorting requires *n* log *n* operations.

Is it possible to do better? Try it. You will find that everyday experience is no help.

We need principles to follow in such cases.

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A	50	**30**	10	40	20	

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Three principles that can help are:

☞ **Consider generalizing a problem when designing an algorithm.**

☞ **Consider Divide and Conquer when designing an algorithm.**

☞ **Consider recursion when designing an algorithm.**

We will use them to derive:

- An Average-Case Linear-Time Median Algorithm
- A Worst-Case Linear-Time Median Algorithm

It is astounding that it is possible to find the median of an unordered array of length *n* in linear time, i.e., time proportional to *n*.

The *median* of an ordered array of n values is the middle value. If n is odd, this is `A[n/2]`; if n is even, we opt for `A[n/2]` rather than averaging the middle two values.

☞ **Consider generalizing a problem when designing an algorithm.**

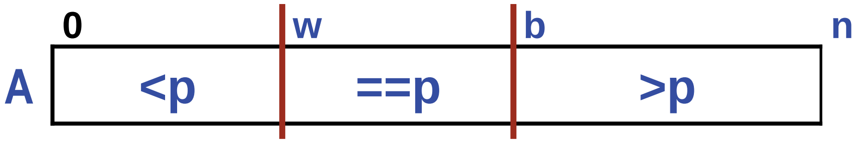
Selection: Given a set of n rank-ordered values, select the $\mathtt{j}^{th}$ smallest value of the set.

Average-Case Linear-Time Algorithm

Selection: Given a set of *n* rank-ordered values, select the j^{th} smallest value of the set.

☞ **Consider Divide and Conquer when designing an algorithm.**

Recall: Partitioning, based on the Dutch National Flag problem, for some pivot *p*:



$0 \le j < w$. The j^{th} smallest value is the j^{th} smallest value of `A[0..w-1]`
$w \le j < b$. The j^{th} smallest value is the pivot, *p*
$b \le j < n$. The j^{th} smallest value is the $(j-b)^{th}$ smallest value in `A[b..n-1]`

Choose one of the three regions based on a Partition (Divide) and repeat (Conquer).

Start with the code for `Partition`, and morph it into `QuickSelect`:

```
/* Rearrange A[L..R-1] into all <p, then all ==p, then all >p. */
static void Partition( int A[], int L, int R, int p ) {
   ⟨body of Partition⟩
   } /* Partition */
```

☞ **Don’t type if you can avoid it; clone. Cut and paste, then adapt.**

Start with the code for `Partition`, and morph it into `QuickSelect`:

```
/* Given int 0≤j<n, return j-th smallest in A[0..n-1]. */
static void QuickSelect( int A[], int L, int R, int p ) {
   ⟨body of Partition⟩
   } /* Partition */
```

☞ **Don't type if you can avoid it; clone. Cut and paste, then adapt.**

Change the name and header comment.

Start with the code for `Partition`, and morph it into `Select`:

```
/* Given int 0≤j<n, return j-th smallest in A[0..n-1]. */
static void QuickSelect( int A[], int n, int j ) {
   int L = 0; int R = n;
   int p = /* value of pivot */ ;
   ⟨body of Partition⟩
   } /* QuickSelect */
```

☞ **Don’t type if you can avoid it; clone. Cut and paste, then adapt.**

Move parameters L, R, and p into the body of QuickSelect, and introduce parameters n and `j`.

Start with the code for `Partition`, and morph it into `Select`:

```
/* Given int 0≤j<n, return j-th smallest in A[0..n-1]. */
static int QuickSelect( int A[], int n, int j ) {
   int L = 0; int R = n;
   int p = /* value of pivot */ ;
   ⟨body of Partition⟩
   return ____;
   } /* QuickSelect */
```

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Change return type to **int**, and introduce a **return** statement for the result.

Could consider recursion, but it is not needed because we can just …

```
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Average-Case Linear-Time Algorithm

INVARIANT

Update L, R, and p iteratively using the **INVARIANT** shown.

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/* Given int 0≤j<n, return j-th smallest in A[0..n-1]. */
static int QuickSelect( int A[], int n, int j ) {
   int L = 0; int R = n;
   int p = /* value of pivot */ ;
   ⟨body of Partition⟩
   return ____;
   } /* QuickSelect */
```
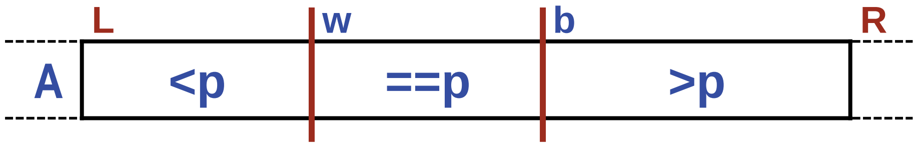

```
/* Initialize. */
while ( /* not finished */ ) {
   /* Compute. */
   /* Go on to next. */
   }
```

Average-Case Linear-Time Algorithm



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      /* Go on to next. */
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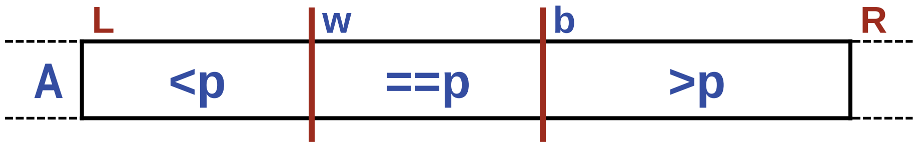
Average-Case Linear-Time Algorithm



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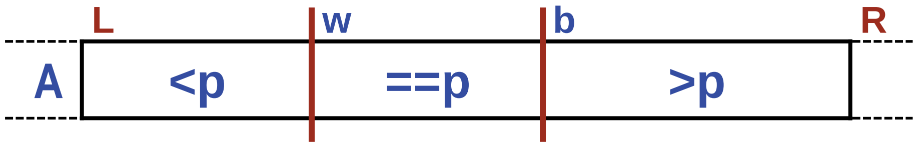
Average-Case Linear-Time Algorithm



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            else if ( j<b ) return p;
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      }
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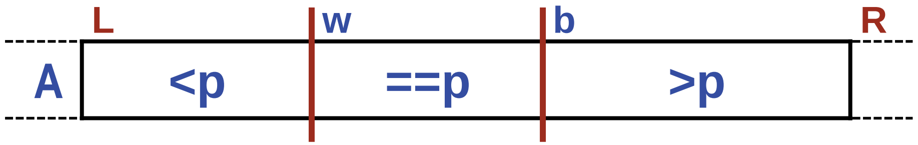

Update L, R, and p iteratively using the **INVARIANT** shown.

```
/* Given int 0≤j<n, return j-th smallest in A[0..n-1]. */
static int QuickSelect( int A[], int n, int j ) {
   int L = 0; int R = n;
   while ( R-L > 1 ) {
      int p = /* value of pivot */ ;
      ⟨body of Partition⟩
      /* Go on to “<p” or “>p” region if j-th smallest there; else return p. */
         if ( j<w ) R = w;
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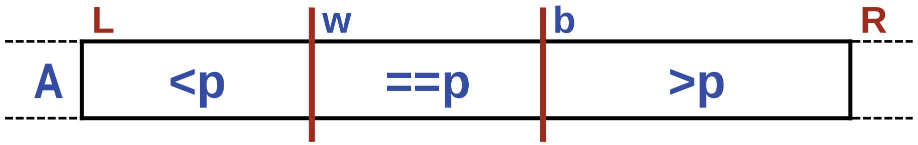

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            else if ( j<b ) return p;
            else L = b;
      }
   return A[j];
   } /* QuickSelect */
```
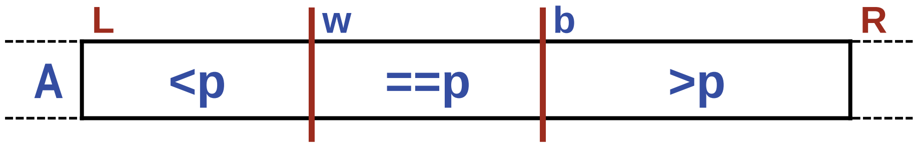

Q. Where was j ever updated?

A. Nowhere. Partitioning moved values so the j^{th} smallest ended up in A[j].

Average-Case Linear-Time Algorithm



Update L, R, and p iteratively using the **INVARIANT** shown.

```
/* Given int 0≤j<n, return j-th smallest in A[0..n-1]. */
static int QuickSelect( int A[], int n, int j ) {
   int L = 0; int R = n;
   while ( R-L > 1 ) {
      int p = (A[L]+A[R-1])/2;
      ⟨body of Partition⟩
      /* Go on to “<p” or “>p” region if j-th smallest there; else return p. */
         if ( j<w ) R = w;
            else if ( j<b ) return p;
            else L = b;
      }
   return A[j];
   } /* QuickSelect */
```

Performance: Pivots computed as (A[L]+A[R-1])/2

- Best case. On each iteration, pivot is (serendipitously) the median of `A[L..R-1]`, so region sizes reduced by ½. Partitioning time is linear in size.

  Total effort. $1 \cdot n + \frac{1}{2} \cdot n + \frac{1}{4} \cdot n + \ldots = 2 \cdot n$, i.e., linear in n

- Worst case. On each iteration, pivot is (serendipitously) the min or max of `A[L..R-1]`, so region sizes reduced by 1. Partitioning time is linear in size.

  Total effort. $n + (n-1) + (n-2) + \ldots + 1 = n \cdot (n-1)/2$, i.e., quadratic in n.

- Average case, i.e., summed over all permutations of values in `A[0..n-1]`.

  Total effort. Linear in n

  (offered without proof)

Worst-Case Linear-Time Algorithm

Bad News: QuickSelect can have quadratic-time performance on some arrays.

Imagine telling the widow:

> But Mrs. Jones, <u>on average</u> the code would have been fast enough to have saved your husband's life.

Goal. Linear-time performance on every array.

Performance Goal: Pivots computed as _________ in the hope that

- Every case.

(1) On each iteration, region sizes reduced by constant ratio r.

Partitioning time is linear in region size.

Total effort for partitioning. $1 \cdot n + r \cdot n + r^2 \cdot n + r^3 \cdot n + \ldots = n/(1-r)$

I.e., linear in *n, not counting time to compute the pivot.*

(2) On each iteration, the cost to compute the pivot is also linear in region size.

Thus, total effort, would be linear in n. In particular, even in the worst-case.

Performance Goal: Pivots computed as approximations to median of `A[L..R-1]`.

- Every case.

  (1) On each iteration, region sizes reduced by constant ratio r.

  Partitioning time is linear in region size.

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Thus, total effort, would be linear in n. In particular, even in the worst-case.

Idea: Pivots computed as approximations to median of `A[L..R-1]`.

Imagine that this array, with median 61:

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
A	51	60	73	92	57	54	75	59	91	58	71	62	67	66	59	52	**61**	72	55	60	79

were laid out in a 3-high 2-D array in row major order:

51	60	73	92	57	54	75
59	91	58	71	62	67	66
59	52	**61**	72	55	60	79

The median of each column is shown in red.

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51	52	58	71	55	54	66
59	60	**61**	72	57	60	75
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Now, imagine that each column were sorted, so its median comes to middle row.

The median of each column is shown in red.

Idea: Pivots computed as approximations to median of `A[L..R-1]`.

Imagine that this array, with median 61:

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
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were laid out in a 3-high 2-D array in row major order:

55	51	52	54	58	66	71
57	59	60	60	**61**	75	72
62	59	91	67	73	79	92

Next, imagine that the columns were sorted by their medians. The median of the medians is shown with a green background.

The median of each column is shown in red.

Idea: Pivots computed as approximations to median of `A[L..R-1]`.

Imagine that this array, with median 61:

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Finally, color code the values:
pink, if ≤ median of medians
blue, if ≥ median of medians

The median of each column is shown in red.

Worst-Case Linear-Time Algorithm

Idea: Pivots computed as approximations to median of `A[L..R-1]`.

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	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
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Choose the median of medians (60) as the pivot p, and partition A.

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
A	51	55	57	54	52	59	59	58	60	60	62	67	66	71	91	**61**	72	75	92	79	73

Idea: Pivots computed as approximations to median of `A[L..R-1]`.

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We seek the median, i.e., the $n/2^{th}$ smallest ($n/2 = 21/2 = 10$), which falls into >p region

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Idea: Pivots computed as approximations to median of `A[L..R-1]`.

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were laid out in a 3-high 2-D array in row major order:

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We seek the median, i.e., the $n/2^{th}$ smallest ($n/2 = 21/2 = 10$), which falls into >p region, eliminating at least $2/3 \cdot ½ = 1/3$ the values.

Thus, the >p region is no larger than r = 1-1/3 = 2/3 the size of the whole.

	0	1	2	3	4	5	6	7	8	9	**10**	11	12	13	14	15	16	17	18	19	20
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Performance Goal: Pivots computed as approximations to median of `A[L..R-1]`.

- Every case.

  (1) On each iteration, region sizes reduced by constant ratio r.

  Partitioning time is linear in region size.

  Total effort for partitioning. $n + r \cdot n + r^2 \cdot n + r^3 \cdot n + \ldots = n/(1-r)$

  I.e., linear in n, *not counting time to compute the pivot.*

  (2) On each iteration, the cost to compute the pivot is also linear in region size.

Thus, total effort, would be linear in n. In particular, even in the worst-case.

Performance Goal: Pivots computed as median of medians of `A[L..R-1]`.

- Every case.

(1) On each iteration, region sizes reduced by constant ratio *r*.

Partitioning time is linear in region size.

Total effort for partitioning. $n + (2/3) \cdot n + (2/3)^2 \cdot n + (2/3)^3 \cdot n + \ldots = n/(1-2/3) = 3 \cdot n$

I.e., linear in *n, not counting time to compute the pivot.*

(2) On each iteration, the cost to compute the pivot is also linear in region size.

Thus, total effort, would be linear in *n*. In particular, even in the worst-case.

Worst-Case Linear-Time Algorithm

Performance Goal: Pivots computed as median of medians of `A[L..R-1]`.

- Every case.

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  I.e., linear in *n, not counting time to compute the pivot.*

  (2) On each iteration, the cost to compute the pivot is also linear in region size.

  But how will we compute the median of medians of `A[L..R-1]`?

  Thus, total effort, would be linear in *n*. In particular, even in the worst-case.

Performance Goal: Pivots computed as median of medians of `A[L..R-1]` using recursion, i.e., apply the worst-case median algorithm to the n/3 medians of groups of 3 elements.

☞ **Consider recursion when designing an algorithm.**

Performance Goal: Pivots computed as median of medians of `A[L..R-1]` using recursion, i.e., apply the worst-case median algorithm to the n/3 medians of groups of 3 elements.

This works, but alas, there are too many groups of 3, so the total cost is super- linear.

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This works, but alas, there are too many groups of 3, so the total cost is super- linear.

But don’t loose heart. All is not lost, because …

Performance Goal: Pivots computed as median of medians of `A[L..R-1]` using recursion, i.e., apply the worst-case median algorithm to the n/5 medians of groups of 5 elements.

This works, and is linear.

Selection of a partition region eliminates at least 3/5 · ½ = 3/10 the values.

Thus, the selected region is no larger than r = 1-3/10 = 7/10 the size of the whole.

Total effort for partitioning. $n + (7/10)\cdot n + (7/10)^2\cdot n + (7/10)^3\cdot n + \ldots = n/(1-7/10) = 3.33\cdot n$

In effect, the reduction ratio r shrinks slightly (from 2/3 to 3/10), but the number of groups shrinks more than enough (from $n/3$ to $n/5$) to render the total linear.