# CS 4110

# Programming Languages & Logics

## Lecture 22
## Polymorphism

21 October 2016

# Announcements

- Adrian will be back Monday, with guest lecturer Yaron Minsky

# Roadmap

Over the last few lectures, we've developed a simple type system for $\lambda$-calculus, extensions for handling a number of language features, and we proved normalization.

Today we'll develop a substantial extension of the simply-typed $\lambda$-calculus by making the type system polymorphic.

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- *Parametric polymorphism* refers to code that is written without knowledge of the actual type of the arguments; the code is parametric in the type of the parameters.

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$$\text{doubleInt} \triangleq \lambda f{:}\,\mathbf{int} \to \mathbf{int}.\ \lambda x{:}\,\mathbf{int}.\ f\,(f\,x)$$

Now suppose we want the same function for booleans, or functions...

$$\begin{aligned}
\text{doubleBool} &\triangleq \lambda f{:}\,\mathbf{bool} \to \mathbf{bool}.\ \lambda x{:}\,\mathbf{bool}.\ f\,(f\,x) \\
\text{doubleFn} &\triangleq \lambda f{:}\,(\mathbf{int} \to \mathbf{int}) \to (\mathbf{int} \to \mathbf{int}).\ \lambda x{:}\,\mathbf{int} \to \mathbf{int}.\ f\,(f\,x) \\
&\ \vdots
\end{aligned}$$

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Every major piece of functionality in a program should be implemented in just one place in the code. When similar functionality is provided by distinct pieces of code, the two should be combined into one by abstracting out the varying parts.

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## Definition (Abstraction Principle)

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In the doubling functions, the varying parts are the types.

We need a way to abstract out the type of the doubling operation, and later instantiate it with different concrete types.

# Polymorphic $\lambda$-Calculus

Invented indepedently in 1972-1974 by a computer scientist John Reynolds and a logician Jean-Yves Girard (who called it System F).

Commonly used as a basis for studying type system extensions

Key feature: function abstraction and application at the type level!

Notation:

- $\Lambda X.\, e$: type abstraction
- $e[\tau]$: type application

Example:
$\lambda X.\, \lambda x : X.\, x$

# Polymorphic $\lambda$-Calculus

## Syntax

$$
\begin{aligned}
e &::= n \mid x \mid \lambda x\!:\!\tau.\, e \mid e_1\, e_2 \mid \Lambda X.\, e \mid e\,[\tau] \\
v &::= n \mid \lambda x\!:\!\tau.\, e \mid \Lambda X.\, e
\end{aligned}
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**Typing Judgment:** $\Delta, \Gamma \vdash e : \tau$

- $\Gamma$ a mapping from variables to types
- $\Delta$ a set of types in scope
- $e$ an expression
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In the polymorphic $\lambda$-calculus, however, we can type this expression using a polymorphic type:

$$\vdash \quad \lambda x : \forall X.\, X \to X.\, x\, [\forall X.\, X \to X]\, x \;:\; (\forall X.\, X \to X) \to (\forall X.\, X \to X)$$

However, all expressions in polymorphic $\lambda$-calculus still halt

# Example: Products

We can encode products in polymorphic $\lambda$-calculus without adding any additional types!

The encodings are based on the (untyped) Church encodings:

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\mathrm{void} \triangleq \forall R.\, R
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erase(e_1\; e_2) & = & erase(e_1)\; erase(e_2) & \\
erase(\Lambda X.\, e) & = & \lambda z.\, erase(e) & \quad \text{where } z \text{ is fresh for } e
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\mathit{erase}(e\; [\tau]) & = & \mathit{erase}(e)\; (\lambda x.\, x) &
\end{array}
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# Type Erasure

The following theorem states this translation is adequate:

**Theorem (Erasure Adequacy)**

*For all expressions $e$ and $e'$, we have $e \rightarrow e'$ iff $\mathit{erase}(e) \rightarrow \mathit{erase}(e')$.*

# Type Inference

The type inference (or "type reconstruction") problem asks whether, for a given untyped $\lambda$-calculus expression $e'$ there exists a well-typed System F expression $e$ such that $\mathit{erase}(e) = e'$

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See Chapter 23 of Pierce for further discussion, as well as restrictions for which type reconstruction is decidable.