CS 4110

Programming Languages & Logics

Lecture 17
Definitional Translation & Continuations

3 October 2016

Announcements

- Homework 4 returned: out of 41, $\bar{x} = 35.3$, $\sigma = 5.3$, median 36
- Wednesday: Prelim I
- My office hours today: 1–2pm, instead of right after class

Products and Let

Syntax

$$
\begin{aligned}
e ::= {} & x \\
\mid {} & \lambda x.\, e \\
\mid {} & e_1\ e_2 \\
\mid {} & (e_1, e_2) \\
\mid {} & \#1\ e \\
\mid {} & \#2\ e \\
\mid {} & \mathsf{let}\ x = e_1\ \mathsf{in}\ e_2
\end{aligned}
$$

$$
\begin{aligned}
v ::= {} & \lambda x.\, e \\
\mid {} & (v_1, v_2)
\end{aligned}
$$

Products and Let

Evaluation Contexts

$$
\begin{aligned}
E ::= {} & [\cdot] \\
\mid {} & E\, e \\
\mid {} & v\, E \\
\mid {} & (E, e) \\
\mid {} & (v, E) \\
\mid {} & \#1\, E \\
\mid {} & \#2\, E \\
\mid {} & \mathsf{let}\, x = E\, \mathsf{in}\, e_2
\end{aligned}
$$

Products and Let

Semantics

$$\frac{e \to e'}{E[e] \to E[e']}$$

$$\frac{}{(\lambda x.\, e)\; v \to e\{v/x\}}\;\beta$$

$$\frac{}{\#1\,(v_1, v_2) \to v_1} \qquad \frac{}{\#2\,(v_1, v_2) \to v_2}$$

$$\frac{}{\mathsf{let}\; x = v\; \mathsf{in}\; e \to e\{v/x\}}$$

Products and Let

Translation

$$
\begin{aligned}
\mathcal{T}[\![x]\!] &= x \\
\mathcal{T}[\![\lambda x.\, e]\!] &= \lambda x.\, \mathcal{T}[\![e]\!] \\
\mathcal{T}[\![e_1\ e_2]\!] &= \mathcal{T}[\![e_1]\!]\ \mathcal{T}[\![e_2]\!] \\
\mathcal{T}[\![(e_1, e_2)]\!] &= (\lambda x.\, \lambda y.\, \lambda f.\, f\, x\, y)\ \mathcal{T}[\![e_1]\!]\ \mathcal{T}[\![e_2]\!] \\
\mathcal{T}[\![\#1\ e]\!] &= \mathcal{T}[\![e]\!]\ (\lambda x.\, \lambda y.\, x) \\
\mathcal{T}[\![\#2\ e]\!] &= \mathcal{T}[\![e]\!]\ (\lambda x.\, \lambda y.\, y) \\
\mathcal{T}[\![\mathsf{let}\ x = e_1\ \mathsf{in}\ e_2]\!] &= (\lambda x.\, \mathcal{T}[\![e_2]\!])\ \mathcal{T}[\![e_1]\!]
\end{aligned}
$$

Laziness

Consider the call-by-name λ-calculus...

Syntax

$$
\begin{aligned}
e &::= x \\
&\mid e_1\ e_2 \\
&\mid \lambda x.\ e
\end{aligned}
$$

$$v ::= \lambda x.\ e$$

Semantics

$$\frac{e_1 \rightarrow e_1'}{e_1\ e_2 \rightarrow e_1'\ e_2} \qquad \frac{}{(\lambda x.\ e_1)\ e_2 \rightarrow e_1\{e_2/x\}}\ \beta$$

Laziness

Translation

$$
\begin{aligned}
\mathcal{T}[\![x]\!] &= x\,(\lambda y.\, y) \\
\mathcal{T}[\![\lambda x.\, e]\!] &= \lambda x.\, \mathcal{T}[\![e]\!] \\
\mathcal{T}[\![e_1\; e_2]\!] &= \mathcal{T}[\![e_1]\!]\,(\lambda z.\, \mathcal{T}[\![e_2]\!]) \qquad z \text{ is not a free variable of } e_2
\end{aligned}
$$

References

Syntax

$$
\begin{aligned}
e ::= {} & x \\
\mid {} & \lambda x.\, e \\
\mid {} & e_0\; e_1
\end{aligned}
$$

$$v ::= \lambda x.\, e$$

References

Syntax

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\begin{aligned}
e ::= {} & x \\
\mid {} & \lambda x.\, e \\
\mid {} & e_0\ e_1 \\
\mid {} & \mathsf{ref}\ e
\end{aligned}
$$

$$v ::= \lambda x.\, e$$

References

Syntax

$$
\begin{aligned}
e ::= {} & x \\
\mid {} & \lambda x.\, e \\
\mid {} & e_0\ e_1 \\
\mid {} & \mathsf{ref}\ e \\
\mid {} & !e
\end{aligned}
$$

$$v ::= \lambda x.\, e$$

References

Syntax

$$
\begin{aligned}
e ::= {} & x \\
\mid {} & \lambda x.\, e \\
\mid {} & e_0\ e_1 \\
\mid {} & \mathsf{ref}\ e \\
\mid {} & \,!e \\
\mid {} & e_1 := e_2
\end{aligned}
$$

$$
v ::= \lambda x.\, e
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References

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\begin{aligned}
E ::= \; & [\cdot] \\
\mid \; & E \; e \\
\mid \; & v \; E \\
\mid \; & \mathsf{ref} \; E \\
\mid \; & !E
\end{aligned}
$$

Evaluation Contexts

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E ::= & \ [\cdot] \\
& \mid E\ e \\
& \mid v\ E \\
& \mid \text{ref}\ E \\
& \mid\ !E \\
& \mid E := e
\end{aligned}
$$

References

Evaluation Contexts

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E ::= {} & [\cdot] \\
\mid {} & E\ e \\
\mid {} & v\ E \\
\mid {} & \text{ref}\ E \\
\mid {} & !E \\
\mid {} & E := e \\
\mid {} & v := E
\end{aligned}
$$

References

Semantics

$$\frac{\langle \sigma, e \rangle \to \langle \sigma', e' \rangle}{\langle \sigma, E[e] \rangle \to \langle \sigma', E[e'] \rangle} \qquad \frac{}{\langle \sigma, (\lambda x.\, e)\; v \rangle \to \langle \sigma, e\{v/x\} \rangle}\;\beta$$

$$\frac{\ell \notin dom(\sigma)}{\langle \sigma, \mathsf{ref}\; v \rangle \to \langle \sigma[\ell \mapsto v], \ell \rangle} \qquad \frac{\sigma(\ell) = v}{\langle \sigma,\; !\ell \rangle \to \langle \sigma, v \rangle}$$

$$\frac{}{\langle \sigma, \ell := v \rangle \to \langle \sigma[\ell \mapsto v], v \rangle}$$

References

Translation

...left as an exercise to the reader. ;-)

Adequacy

How do we know if a translation is correct?

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Every target evaluation should represent a source evaluation...

Definition (Soundness)

$\forall e \in \mathbf{Exp}_{\text{src}}.$ if $\mathcal{T}[\![e]\!] \rightarrow^*_{\text{trg}} v'$ then $\exists v.\ e \rightarrow^*_{\text{src}} v$
and v' equivalent to v

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...and every source evaluation should have a target evaluation:

Definition (Completeness)

$\forall e \in \mathbf{Exp}_{\mathrm{src}}.$ if $e \rightarrow^*_{\mathrm{src}} v$ then $\exists v'.\ \mathcal{T}[\![e]\!] \rightarrow^*_{\mathrm{trg}} v'$ and v' equivalent to v

Continuations

In the preceding translations, the control structure of the source language was translated directly into the corresponding control structure in the target language.

For example:

$$\begin{aligned}\mathcal{T}[\![\lambda x.\, e]\!] &= \lambda x.\, \mathcal{T}[\![e]\!] \\ \mathcal{T}[\![e_1\ e_2]\!] &= \mathcal{T}[\![e_1]\!]\ \mathcal{T}[\![e_2]\!]\end{aligned}$$

What can go wrong with this approach?

Continuations

- A snippet of code that represents “the rest of the program”
- Can be used directly by programmers...
- ...or in program transformations by a compiler
- Make the control flow of the program explicit
- Also useful for defining the meaning of features like exceptions

Example

Consider the following expression:

$$(\lambda x.\, x)\ ((1 + 2) + 3) + 4$$

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k_3 &= \lambda c.\, k_2\, (c + 2)
\end{aligned}$$

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k_2 &= \lambda b.\, k_1\, (b + 3) \\
k_3 &= \lambda c.\, k_2\, (c + 2)
\end{aligned}$$

The original expression is equivalent to $k_3\ 1$, or:

$$(\lambda c.\, (\lambda b.\, (\lambda a.\, (\lambda v.\, (\lambda x.\, x)\, v)\, (a + 4))\, (b + 3))\, (c + 2))\, 1$$

Example (Continued)

Recall that let $x = e$ in e' is syntactic sugar for $(\lambda x.\, e')\; e$.

Hence, we can rewrite the expression with continuations more succinctly as

$$\begin{array}{l}
\text{let } c = 1 \text{ in} \\
\text{let } b = c + 2 \text{ in} \\
\text{let } a = b + 3 \text{ in} \\
\text{let } v = a + 4 \text{ in} \\
(\lambda x.\, x)\; v
\end{array}$$

CPS Transformation

We write $\mathcal{CPS}[\![e]\!]\,k = \ldots$ instead of $\mathcal{CPS}[\![e]\!] = \lambda k.\ \ldots$

We assume that the new variables introduced are "fresh".

CPS Transformation

$$\mathcal{CPS}[\![n]\!]\,k = k\,n$$

We write $\mathcal{CPS}[\![e]\!]\,k = \ldots$ instead of $\mathcal{CPS}[\![e]\!] = \lambda k.\ \ldots$

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CPS Transformation

$$\begin{aligned} \mathcal{CPS}[\![n]\!]\,k &= k\,n \\ \mathcal{CPS}[\![e_1 + e_2]\!]\,k &= \mathcal{CPS}[\![e_1]\!]\,(\lambda n.\,\mathcal{CPS}[\![e_2]\!]\,(\lambda m.\,k\,(n+m))) \end{aligned}$$

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\mathcal{CPS}[\![\#1\;e]\!]\,k &= \mathcal{CPS}[\![e]\!]\,(\lambda v.\, k\,(\#1\;v))
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\mathcal{CPS}[\![x]\!]\,k &= k\,x \\
\mathcal{CPS}[\![\lambda x.\,e]\!]\,k &= k\,(\lambda x.\,\lambda k'.\,\mathcal{CPS}[\![e]\!]\,k')
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