CS 4110

Programming Languages & Logics

Lecture 16
Programming in the λ-calculus

30 September 2016

Review: Church Booleans

We can encode TRUE, FALSE, and IF, as:

$$\begin{aligned} \text{TRUE} &\triangleq \lambda x.\, \lambda y.\, x \\ \text{FALSE} &\triangleq \lambda x.\, \lambda y.\, y \\ \text{IF} &\triangleq \lambda b.\, \lambda t.\, \lambda f.\, b\ t\ f \end{aligned}$$

This way, IF behaves how it ought to:

$$\begin{aligned} \text{IF TRUE}\ v_t\ v_f &\rightarrow^* v_t \\ \text{IF FALSE}\ v_t\ v_f &\rightarrow^* v_f \end{aligned}$$

Review: Church Numerals

Church numerals encode a number n as a function that takes f and x, and applies f to x n times.

$$\begin{aligned}
\overline{0} &\triangleq \lambda f.\, \lambda x.\, x \\
\overline{1} &\triangleq \lambda f.\, \lambda x.\, f\, x \\
\overline{2} &\triangleq \lambda f.\, \lambda x.\, f\,(f\, x)
\end{aligned}$$

We can define other functions on integers:

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\mathrm{TIMES} &\triangleq \lambda n_1.\, \lambda n_2.\, n_1\, (\mathrm{PLUS}\ \mathrm{n}_2)\, \overline{0} \\
\mathrm{ISZERO} &\triangleq \lambda n.\, n\, (\lambda z.\, \mathrm{FALSE})\, \mathrm{TRUE}
\end{aligned}
$$

Recursive Functions

How would we write recursive functions like factorial?

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We'd like to write it like this...

$$\mathsf{FACT} \triangleq \lambda n.\, \mathsf{IF}\ (\mathsf{ISZERO}\ n)\ 1\ (\mathsf{TIMES}\ n\ (\mathsf{FACT}\ (\mathsf{PRED}\ n)))$$

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$$\text{FACT} \triangleq \lambda n.\, \text{IF}\ (\text{ISZERO}\ n)\ 1\ (\text{TIMES}\ n\ (\text{FACT}\ (\text{PRED}\ n)))$$

In slightly more readable notation this is...

$$\text{FACT} \triangleq \lambda n.\, \textbf{if}\ n = 0\ \textbf{then}\ 1\ \textbf{else}\ n \times \text{FACT}\ (n-1)$$

...but this is an equation, not a definition!

Recursion removal trick

We can perform a “trick” to define a function FACT that satisfies the recursive equation on the previous slide.

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Define a new function FACT′:

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Define a new function FACT′:

$$\mathrm{FACT}' \triangleq \lambda f.\, \lambda n.\, \textbf{if}\ n = 0\ \textbf{then}\ 1\ \textbf{else}\ n \times (f\, f\, (n - 1))$$

Then define FACT as FACT′ applied to itself:

$$\mathrm{FACT} \triangleq \mathrm{FACT}'\ \mathrm{FACT}'$$

Example

Let’s try evaluating FACT on 3...

FACT 3

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$$\text{FACT } 3 = (\text{FACT}' \ \text{FACT}') \ 3$$

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$$\begin{aligned}\mathsf{FACT}\ 3 &= (\mathsf{FACT}'\ \mathsf{FACT}')\ 3\\ &= ((\lambda f.\ \lambda n.\ \textbf{if}\ n = 0\ \textbf{then}\ 1\ \textbf{else}\ n \times (f\ f\ (n-1)))\ \mathsf{FACT}')\ 3\end{aligned}$$

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&\to (\lambda n.\, \textbf{if}\ n = 0\ \textbf{then}\ 1\ \textbf{else}\ n \times (\text{FACT}'\ \text{FACT}'\ (n-1)))\ 3
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&\rightarrow 3 \times (\mathsf{FACT}'\ \mathsf{FACT}'\ (3-1))
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&= 3 \times (\mathsf{FACT}\,(3-1))
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&\to \ldots \\
&\to 3 \times 2 \times 1 \times 1
\end{aligned}
$$

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&\to \ldots \\
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&\to \ldots \\
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\end{aligned}$$

So we have a technique for writing recursive functions: write a function f' that takes itself as an argument and define f as $f'\ f'$.

Fixed point combinators

There is another way: fixed points!

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Consider factorial again. It is a fixed point of the following:

$$G \triangleq \lambda f.\, \lambda n.\, \textbf{if}\ n = 0\ \textbf{then}\ 1\ \textbf{else}\ n \times (f\,(n - 1))$$

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Recall that if g if a fixed point of G, then $G\,g = g$. To see that any fixed point g is a real factorial function, try evaluating it:

$$g\,5$$

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$$\begin{aligned} g\,5 &= (G\,g)\,5 \\ &\to^{*} 5 \times (g\,4) \\ &= 5 \times ((G\,g)\,4) \end{aligned}$$

Fixed point combinators

How can we generate the fixed point of G?

In denotational semantics, finding fixed points took a lot of math. In the λ-calculus, we just need a suitable combinator...

Y Combinator

The (infamous) Y combinator is defined as

$$\mathsf{Y} \triangleq \lambda f.\, (\lambda x.\, f\, (x\, x))\, (\lambda x.\, f\, (x\, x))$$

We say that Y is a *fixed point combinator* because Y f is a fixed point of f (for any lambda term f).

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What happens when we evaluate Y G under CBV?

Z Combinator

To avoid this issue, we'll use a slight variant of the Y combinator, called Z, which is easier to use under CBV.

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$$\mathsf{Z} \triangleq \lambda f.\,(\lambda x.\,f\,(\lambda y.\,x\,x\,y))\,(\lambda x.\,f\,(\lambda y.\,x\,x\,y))$$

Example

Let's see Z in action, on our function G.

FACT

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\end{aligned}
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\begin{array}{rl}
& \text{FACT} \\
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= & (\lambda f.\, (\lambda x.\, f\, (\lambda y.\, x\, x\, y))\, (\lambda x.\, f\, (\lambda y.\, x\, x\, y)))\, G \\
\rightarrow & (\lambda x.\, G\, (\lambda y.\, x\, x\, y))\, (\lambda x.\, G\, (\lambda y.\, x\, x\, y))
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\to & (\lambda x.\, G\,(\lambda y.\, x\,x\,y))\,(\lambda x.\, G\,(\lambda y.\, x\,x\,y)) \\
\to & G\,(\lambda y.\, (\lambda x.\, G\,(\lambda y.\, x\,x\,y))\,(\lambda x.\, G\,(\lambda y.\, x\,x\,y))\, y) \\
= & (\lambda f.\, \lambda n.\, \textbf{if}\ n = 0\ \textbf{then}\ 1\ \textbf{else}\ n \times (f\,(n-1))) \\
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= & \lambda n.\,\textbf{if}\ n = 0\ \textbf{then}\ 1\ \textbf{else}\ n \times (\text{FACT}\,(n-1))
\end{array}
$$

Other fixed point combinators

There are many (indeed infinitely many) fixed-point combinators. Here's a cute one:

$$\mathsf{Y}_k \triangleq (\mathsf{L\,L\,L\,L\,L\,L\,L\,L\,L\,L\,L\,L\,L\,L\,L\,L\,L\,L\,L\,L\,L\,L\,L\,L\,L\,L})$$

where

$$\begin{aligned}\mathsf{L} \triangleq\ & \lambda abcdefghijklmnopqstuvwxyzr. \\ & (r\,(t\,h\,i\,s\,i\,s\,a\,f\,i\,x\,e\,d\,p\,o\,i\,n\,t\,c\,o\,m\,b\,i\,n\,a\,t\,o\,r))\end{aligned}$$

Turing's Fixed Point Combinator

To gain some more intuition for fixed point combinators, let's derive a combinator Θ originally discovered by Turing.

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We know that $\Theta\, f$ is a fixed point of f, so we have

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$$\Theta = \lambda f.\, f\,(\Theta\, f).$$

Now use the recursion removal trick:

$$\begin{aligned} \Theta' &\triangleq \lambda t.\, \lambda f.\, f\,(t\, t\, f) \\ \Theta &\triangleq \Theta'\, \Theta' \end{aligned}$$

θ Example

$\text{FACT} = \Theta\ G$

θ Example

$$
\begin{aligned}
\mathsf{FACT} &= \Theta\, G \\
&= ((\lambda t.\, \lambda f.\, f\,(t\,t\,f))\,(\lambda t.\, \lambda f.\, f\,(t\,t\,f)))\, G
\end{aligned}
$$

θ Example

$$\begin{aligned}
\mathsf{FACT} &= \Theta\, G \\
&= ((\lambda t.\, \lambda f.\, f\, (t\, t\, f))\, (\lambda t.\, \lambda f.\, f\, (t\, t\, f)))\, G \\
&\rightarrow (\lambda f.\, f\, ((\lambda t.\, \lambda f.\, f\, (t\, t\, f))\, (\lambda t.\, \lambda f.\, f\, (t\, t\, f))\, f))\, G
\end{aligned}$$

θ Example

$$
\begin{aligned}
\mathsf{FACT} &= \Theta\, G \\
&= ((\lambda t.\, \lambda f.\, f\,(t\,t\,f))\,(\lambda t.\, \lambda f.\, f\,(t\,t\,f)))\, G \\
&\to (\lambda f.\, f\,((\lambda t.\, \lambda f.\, f\,(t\,t\,f))\,(\lambda t.\, \lambda f.\, f\,(t\,t\,f))\, f))\, G \\
&\to G\,((\lambda t.\, \lambda f.\, f\,(t\,t\,f))\,(\lambda t.\, \lambda f.\, f\,(t\,t\,f))\, G)
\end{aligned}
$$

θ Example

$$
\begin{aligned}
\mathsf{FACT} &= \Theta\, G \\
&= ((\lambda t.\, \lambda f.\, f\,(t\,t\,f))\,(\lambda t.\, \lambda f.\, f\,(t\,t\,f)))\,G \\
&\to (\lambda f.\, f\,((\lambda t.\, \lambda f.\, f\,(t\,t\,f))\,(\lambda t.\, \lambda f.\, f\,(t\,t\,f))\,f))\,G \\
&\to G\,((\lambda t.\, \lambda f.\, f\,(t\,t\,f))\,(\lambda t.\, \lambda f.\, f\,(t\,t\,f))\,G) \\
&= G\,(\Theta\, G)
\end{aligned}
$$

θ Example

$$
\begin{aligned}
\text{FACT} &= \Theta\, G \\
&= ((\lambda t.\, \lambda f.\, f\,(t\,t\,f))\,(\lambda t.\, \lambda f.\, f\,(t\,t\,f)))\, G \\
&\to (\lambda f.\, f\,((\lambda t.\, \lambda f.\, f\,(t\,t\,f))\,(\lambda t.\, \lambda f.\, f\,(t\,t\,f))\, f))\, G \\
&\to G\,((\lambda t.\, \lambda f.\, f\,(t\,t\,f))\,(\lambda t.\, \lambda f.\, f\,(t\,t\,f))\, G) \\
&= G\,(\Theta\, G) \\
&= (\lambda f.\, \lambda n.\, \textbf{if}\ n = 0\ \textbf{then}\ 1\ \textbf{else}\ n \times (f\,(n-1)))\,(\Theta\, G) \\
&\to \lambda n.\, \textbf{if}\ n = 0\ \textbf{then}\ 1\ \textbf{else}\ n \times ((\Theta\, G)\,(n-1)) \\
&= \lambda n.\, \textbf{if}\ n = 0\ \textbf{then}\ 1\ \textbf{else}\ n \times (\text{FACT}\,(n-1))
\end{aligned}
$$

Definitional Translation

We know how to encode Booleans, conditionals, natural numbers, and recursion in λ-calculus.

Can we define a *real* programming language by translating everything in it into the λ-calculus?

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Can we define a *real* programming language by translating everything in it into the λ-calculus?

In definitional translation, we define a denotational semantics where the target is a simpler programming language instead of mathematical objects.

Review: Call-by-Value

Here are the syntax and CBV semantics of λ-calculus:

$$
\begin{aligned}
e &::= x \mid \lambda x.\, e \mid e_1\, e_2 \\
v &::= \lambda x.\, e
\end{aligned}
$$

$$
\frac{e_1 \rightarrow e_1'}{e_1\, e_2 \rightarrow e_1'\, e_2} \qquad \frac{e \rightarrow e'}{v\, e \rightarrow v\, e'}
$$

$$
\frac{}{(\lambda x.\, e)\, v \rightarrow e\{v/x\}}\ \beta
$$

There are two kinds of rules: *congruence rules* that specify evaluation order and *computation rules* that specify the "interesting" reductions.

Evaluation Contexts

Evaluation contexts let us separate out these two kinds of rules.

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An evaluation context E is an expression with a "hole" in it: a single occurrence of the special symbol $[\cdot]$ in place of a subexpression.

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An evaluation context E is an expression with a "hole" in it: a single occurrence of the special symbol $[\cdot]$ in place of a subexpression.

$$E ::= [\cdot] \mid E\ e \mid v\ E$$

We write $E[e]$ to mean the evaluation context E where the hole has been replaced with the expression e.

Examples

$$E_1 = [\cdot]\,(\lambda x.\, x)$$

$$E_1[\lambda y.\, y\, y] = (\lambda y.\, y\, y)\ \lambda x.\, x$$

Examples

$$E_1 = [\cdot]\,(\lambda x.\,x)$$
$$E_1[\lambda y.\,y\,y] = (\lambda y.\,y\,y)\;\lambda x.\,x$$

$$E_2 = (\lambda z.\,z\,z)\,[\cdot]$$
$$E_2[\lambda x.\,\lambda y.\,x] = (\lambda z.\,z\,z)\,(\lambda x.\,\lambda y.\,x)$$

Examples

$$
\begin{aligned}
E_1 &= [\cdot]\,(\lambda x.\, x) \\
E_1[\lambda y.\, y\, y] &= (\lambda y.\, y\, y)\ \lambda x.\, x
\end{aligned}
$$

$$
\begin{aligned}
E_2 &= (\lambda z.\, z\, z)\ [\cdot] \\
E_2[\lambda x.\, \lambda y.\, x] &= (\lambda z.\, z\, z)\ (\lambda x.\, \lambda y.\, x)
\end{aligned}
$$

$$
\begin{aligned}
E_3 &= ([\cdot]\ \lambda x.\, x\, x)\ ((\lambda y.\, y)\ (\lambda y.\, y)) \\
E_3[\lambda f.\, \lambda g.\, f\, g] &= ((\lambda f.\, \lambda g.\, f\, g)\ \lambda x.\, x\, x)\ ((\lambda y.\, y)\ (\lambda y.\, y))
\end{aligned}
$$

CBV With Evaluation Contexts

With evaluation contexts, we can define the evaluation semantics for the CBV λ-calculus with just two rules: one for evaluation contexts, and one for β-reduction.

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With evaluation contexts, we can define the evaluation semantics for the CBV λ-calculus with just two rules: one for evaluation contexts, and one for β-reduction.

With this syntax:

$$E ::= [\cdot] \mid E\,e \mid v\,E$$

The small-step rules are:

$$\frac{e \rightarrow e'}{E[e] \rightarrow E[e']}$$

$$\frac{}{(\lambda x.\, e)\, v \rightarrow e\{v/x\}}\ \beta$$

CBN With Evaluation Contexts

We can also define the semantics of CBN λ-calculus with evaluation contexts.

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For call-by-name, the syntax for evaluation contexts is different:

$$E ::= [\cdot] \mid E\ e$$

But the small-step rules are the same:

$$\frac{e \to e'}{E[e] \to E[e']}$$

$$\frac{}{(\lambda x.\ e)\ e' \to e\{e'/x\}}\ \beta$$

Multi-Argument λ-calculus

Let's define a version of the λ-calculus that allows functions to take multiple arguments.

$$e ::= x \mid \lambda x_1, \ldots, x_n.\, e \mid e_0\, e_1\ \ldots\ e_n$$

Multi-Argument λ-calculus

We can define a CBV operational semantics:

$$E ::= [\cdot] \mid v_0 \ \ldots \ v_{i-1} \, E \, e_{i+1} \ \ldots \ e_n$$

$$\frac{e \to e'}{E[e] \to E[e']}$$

$$\frac{}{(\lambda x_1, \ldots, x_n.\, e_0)\; v_1 \ \ldots \ v_n \to e_0\{v_1/x_1\}\{v_2/x_2\} \ldots \{v_n/x_n\}}\ \beta$$

The evaluation contexts ensure that we evaluate multi-argument applications $e_0 \; e_1 \ \ldots \ e_n$ from left to right.

Definitional Translation

The multi-argument λ-calculus isn't any more expressive that the pure λ-calculus.

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We can define a translation $\mathcal{T}[\![\cdot]\!]$ that takes an expression in the multi-argument λ-calculus and returns an equivalent expression in the pure λ-calculus.

Definitional Translation

The multi-argument λ-calculus isn't any more expressive that the pure λ-calculus.

We can define a translation $\mathcal{T}[\![\cdot]\!]$ that takes an expression in the multi-argument λ-calculus and returns an equivalent expression in the pure λ-calculus.

$$
\begin{aligned}
\mathcal{T}[\![x]\!] &= x \\
\mathcal{T}[\![\lambda x_1, \ldots, x_n.\, e]\!] &= \lambda x_1.\ \ldots\ \lambda x_n.\, \mathcal{T}[\![e]\!] \\
\mathcal{T}[\![e_0\ e_1\ e_2\ \ldots\ e_n]\!] &= (\ldots((\mathcal{T}[\![e_0]\!]\ \mathcal{T}[\![e_1]\!])\ \mathcal{T}[\![e_2]\!])\ \ldots\ \mathcal{T}[\![e_n]\!])
\end{aligned}
$$

This translation *curries* the multi-argument λ-calculus.