

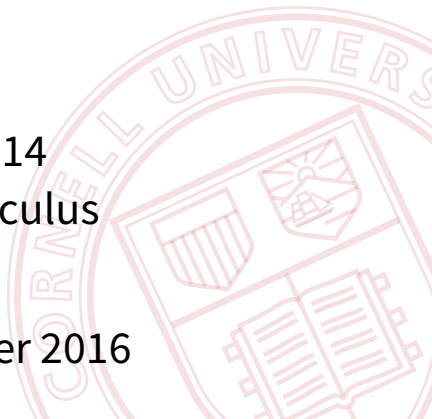
CS 4110

# Programming Languages & Logics

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Lecture 14  
More  $\lambda$ -calculus

26 September 2016



# Announcements

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- Homework #3 returned
  - ▶ Out of 40,  $\bar{x} = 35.9$  and  $\sigma = 8$
- Homework #4 due Wednesday
- Preliminary Exam I **next Wednesday**, October 5
  - ▶ Topics: Up through Hoare logic. (No  $\lambda$ -calculus.)
  - ▶ In class; 50 minutes. (Show up on time to get all 50 minutes.)
  - ▶ Closed book and closed notes.
  - ▶ If the problems use any definitions (the operational semantics for IMP, the Hoare logic proof rules, etc.), those will be provided.
  - ▶ Practice problems now available on CMS.

# Review: $\lambda$ -calculus

## Syntax

$$\begin{aligned} e &::= x \mid e_1 e_2 \mid \lambda x. e \\ v &::= \lambda x. e \end{aligned}$$

## Semantics (call by value)

$$\frac{e_1 \rightarrow e'_1}{e_1 e_2 \rightarrow e'_1 e_2} \quad \frac{e \rightarrow e'}{v e \rightarrow v e'}$$

$$\frac{}{(\lambda x. e) v \rightarrow e\{v/x\}} \beta$$

## Example: Twice

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$$\text{hexadecatuple } x = \text{octuple}(\text{octuple } x)$$



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$$twice \triangleq \lambda f. \lambda x. f(f\ x)$$

## Example: Twice

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Now suppose we want to apply *double* multiple times:

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We can abstract this pattern using a generic function:

$$twice \triangleq \lambda f. \lambda x. f(f\ x)$$

Now the functions above can be written as

$$\begin{aligned} quadruple &= twice\ double \\ octuple &= twice\ quadruple \\ hexadecatuple &= twice\ octuple \\ &\quad (or\ twice\ (\lambda x. twice\ x)) \end{aligned}$$

# Evaluation

The essence of  $\lambda$ -calculus evaluation is the  $\beta$ -reduction rule, which says how to apply a function to an argument.

$$\overline{(\lambda x. e) v \rightarrow e\{v/x\}} \quad \beta\text{-REDUCTION}$$

But there are many different evaluation strategies, each corresponding to particular ways of using  $\beta$ -reduction:

- Call-by-value
- Call-by-name
- “Full”  $\beta$ -reduction
- ...

# Call by value

$$\frac{e_1 \rightarrow e'_1}{e_1 e_2 \rightarrow e'_1 e_2} \qquad \frac{e_2 \rightarrow e'_2}{v_1 e_2 \rightarrow v_1 e'_2}$$

$$\frac{}{(\lambda x. e_1) v_2 \rightarrow e_1 \{v_2/x\}} \beta$$

Key characteristics:

- Arguments evaluated fully before they are supplied to functions
- Evaluation goes from left to right (in this presentation)
- We don't evaluate "under a  $\lambda$ "

# Call by name

$$\frac{e_1 \rightarrow e'_1}{e_1 e_2 \rightarrow e'_1 e_2}$$

$$\frac{}{(\lambda x. e_1) e_2 \rightarrow e_1 \{e_2/x\}} \beta$$

Key characteristics:

- Arguments supplied immediately to functions
- Evaluation still goes from left to right (in this presentation)
- We still don't evaluate "under a  $\lambda$ "

# Full $\beta$ reduction

$$\frac{e_1 \rightarrow e'_1}{e_1 e_2 \rightarrow e'_1 e_2} \quad \frac{e_2 \rightarrow e'_2}{e_1 e_2 \rightarrow e_1 e'_2}$$

$$\frac{e \rightarrow e'}{\lambda x. e \rightarrow \lambda x. e'}$$

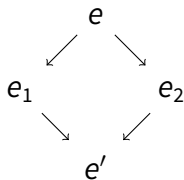
$$\frac{}{(\lambda x. e_1) e_2 \rightarrow e_1 \{e_2/x\}} \beta$$

Key characteristics:

- Use the  $\beta$  rule anywhere...
- ...including “under a  $\lambda$ ”...
- ...nondeterministically.

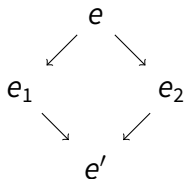
# Confluence

Full  $\beta$  reduction has this property:



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## Theorem (Confluence)

*If  $e \rightarrow^* e_1$  and  $e \rightarrow^* e_2$  then  $e_1 \rightarrow^* e'$  and  $e_2 \rightarrow^* e'$  for some  $e'$ .*



# Substitution

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The main workhorse in the  $\beta$  rule is **substitution**, which replaces free occurrences of a variable  $x$  with a term  $e$ .

However, defining substitution  $e_1\{e_2/x\}$  correctly is tricky...

# “Substitution”

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As a first attempt, consider:

$$y\{e/x\} = \begin{cases} e & \text{if } y = x \\ y & \text{otherwise} \end{cases}$$

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What's wrong with this definition?

It substitutes bound variables too!

$$(\lambda y.y)\{3/y\}$$

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What's wrong with this definition?

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$$(\lambda y.y)\{3/y\} = (\lambda y.3)$$

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Okay... let's avoid rewriting bound variables by relying on  $\alpha$ -equivalence. We'll require that abstractions don't use  $x$ , the variable we're substituting.



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We *assume away* abstractions over  $x$ . (Thanks,  $\alpha$ -equivalence!)

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It leads to variable capture!

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# Real Substitution

The correct definition is *capture-avoiding substitution*:

$$\begin{aligned}y\{e/x\} &= \begin{cases} e & \text{if } y = x \\ y & \text{otherwise} \end{cases} \\(e_1 e_2)\{e/x\} &= (e_1\{e/x\}) (e_2\{e/x\}) \\(\lambda y.e_1)\{e/x\} &= \lambda y.(e_1\{e/x\}) \quad \text{where } y \neq x \text{ and } y \notin \text{fv}(e)\end{aligned}$$

where  $\text{fv}(e)$  is the *free variables* of a term  $e$ .