CS 4110

Programming Languages & Logics

Lecture 2
Introduction to Semantics

26 August 2016

~~29 August 2012~~

Announcements

~~Wednesday Lecture~~

- ~~Moved to Thurston 203~~

Kozen
~~Foster~~ Office Hours

- Today ~~11a–12pm in Gates 432~~ 10-11am Gates 436

~~Mota Office Hours~~

- ~~Wed 11am–12pm in TBD~~
- ~~Thurs 2:30pm–4pm in TBD~~

Homework #1

- Out: Wednesday, ~~September 3rd~~ August 31
- Due: Wednesday, September ~~10th~~ Sept 7
- Distributed via CMS <- most likely

Semantics

Question: What is the meaning of a program?

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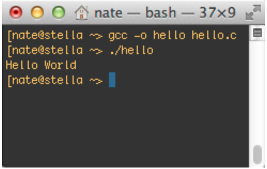
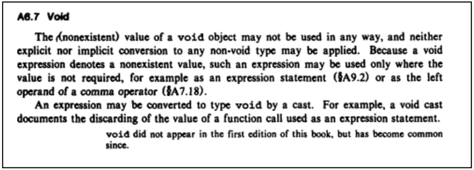
Answer: We could execute the program using an interpreter or a compiler, or we could consult a manual...




A6.7 Void

The (nonexistent) value of a `void` object may not be used in any way, and neither explicit nor implicit conversion to any non-void type may be applied. Because a void expression denotes a nonexistent value, such an expression may be used only where the value is not required, for example as an expression statement (§A9.2) or as the left operand of a comma operator (§A7.18).

An expression may be converted to type `void` by a cast. For example, a void cast documents the discarding of the value of a function call used as an expression statement.

`void` did not appear in the first edition of this book, but has become common since.

...but none of these is a satisfactory solution.

Formal Semantics

Three Approaches

- Operational $\langle \sigma, e \rangle \longrightarrow \langle \sigma', e' \rangle$
  - Model program by execution on abstract machine
  - Useful for implementing compilers and interpreters
- Denotational: $[\![e]\!]$
  - Model program as mathematical objects
  - Useful for theoretical foundations
- Axiomatic $\vdash \{\phi\}\, e \,\{\psi\}$
  - Model program by the logical formulas it obeys
  - Useful for proving program correctness

Arithmetic Expressions

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BNF Grammar:

$$\begin{array}{rl} e ::= & x \\ | & n \\ | & e_1 + e_2 \\ | & e_1 * e_2 \\ | & x := e_1 \,;\, e_2 \end{array}$$

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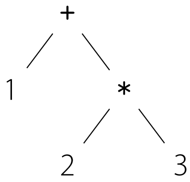

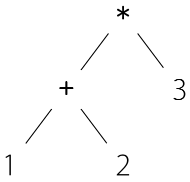

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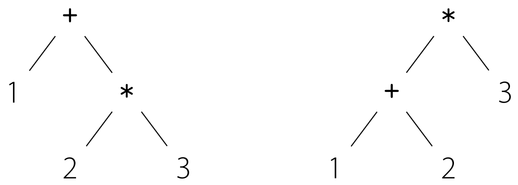

In this course, we will distinguish abstract syntax from concrete syntax, and focus primarily on abstract syntax (using conventions or parentheses at the concrete level to disambiguate as needed).

Representing Expressions

BNF Grammar:

$$
\begin{aligned}
e ::= {} & x \\
\mid {} & n \\
\mid {} & e_1 \mathbin{\texttt{+}} e_2 \\
\mid {} & e_1 \mathbin{\texttt{*}} e_2 \\
\mid {} & x \mathbin{\texttt{:=}} e_1 \texttt{;}\ e_2
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OCaml:

```
type exp = Var of string
         | Int of int
         | Add of exp * exp
         | Mul of exp * exp
         | Assgn of string * exp * exp
```

Example: Mul(Int 2, Add(Var "foo", Int 1))

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Java:

```
abstract class Expr { }
class Var extends Expr { String name; ... }
class Int extends Expr { int val; ... }
class Add extends Expr { Expr exp1, exp2; ... }
class Mul extends Expr { Expr exp1, exp2; ... }
class Assgn extends Expr { String var, Expr exp1, exp2; ... }
```

Example: new Mul(new Int(2), new Add(new Var("foo"), new Int(1)))

Quiz

- $7 + (4 * 2)$ evaluates to ...?

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The rest of this lecture will make these intuitions precise...

Mathematical Preliminaries

Binary Relations

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Some Important Relations

- empty – $\emptyset$
- total – $A \times B$
- identity on A – $\{(a, a) \mid a \in A\}$.
- composition $R; S$ – $\{(a, c) \mid \exists b.\ (a, b) \in R \wedge (b, c) \in S\}$

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The *image* of f is the set of elements $b \in B$ that are mapped to by at least one $a \in A$. More formally: $\text{image}(f) \triangleq \{f(a) \mid a \in A\}$

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A function $f : A \to B$ is said to be *surjective* (or *onto*) if and only if the image of f is B.

Operational Semantics

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More formally,

$$\begin{aligned} \textbf{Store} &\triangleq \textbf{Var} \rightharpoonup \textbf{Int} \\ \textbf{Config} &\triangleq \textbf{Store} \times \textbf{Exp} \end{aligned}$$

Note that a store is a *partial* function from variables to integers.

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Answer: define it inductively, using inference rules:

$$\frac{p = m + n}{\langle \sigma, n \mathbin{\mathbf{+}} m \rangle \longrightarrow \langle \sigma, p \rangle} \text{ Add}$$

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Intuitively, if facts above the line hold, then facts below the line hold. More formally, "$\longrightarrow$" is the smallest relation "closed" under the inference rules.

Variables

$$\frac{n = \sigma(x)}{\langle \sigma, x \rangle \longrightarrow \langle \sigma, n \rangle} \text{ Var}$$

Addition

$$\frac{\langle \sigma, e_1 \rangle \longrightarrow \langle \sigma', e_1' \rangle}{\langle \sigma, e_1 + e_2 \rangle \longrightarrow \langle \sigma', e_1' + e_2 \rangle} \text{ LAdd}$$

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Multiplication

$$\frac{\langle \sigma, e_1 \rangle \longrightarrow \langle \sigma', e_1' \rangle}{\langle \sigma, e_1 * e_2 \rangle \longrightarrow \langle \sigma', e_1' * e_2 \rangle} \text{ LMul}$$

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$$\frac{p = m \times n}{\langle \sigma, m * n \rangle \longrightarrow \langle \sigma, p \rangle} \text{ Mul}$$

Assignment

$$\frac{\langle \sigma, e_1 \rangle \longrightarrow \langle \sigma', e_1' \rangle}{\langle \sigma, x := e_1 \ ; \ e_2 \rangle \longrightarrow \langle \sigma', x := e_1' \ ; \ e_2 \rangle} \text{ Assgn1}$$

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$$\frac{\sigma' = \sigma[x \mapsto n]}{\langle \sigma, x := n \,;\, e_2 \rangle \longrightarrow \langle \sigma', e_2 \rangle}\ \text{Assgn}$$

Notation: $\sigma[x \mapsto n]$ maps x to n and otherwise behaves like σ

Operational Semantics

$$\frac{n = \sigma(x)}{\langle \sigma, x \rangle \rightarrow \langle \sigma, n \rangle} \text{ Var}$$

$$\frac{\langle \sigma, e_1 \rangle \rightarrow \langle \sigma', e_1' \rangle}{\langle \sigma, e_1 + e_2 \rangle \rightarrow \langle \sigma', e_1' + e_2 \rangle} \text{ LAdd}$$

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