

CS 4110

Programming Languages & Logics

Lecture 15

De Bruijn, Combinators, Encodings

28 September 2016



Review: λ -calculus

Syntax

$$\begin{aligned} e &::= x \mid e_1 e_2 \mid \lambda x. e \\ v &::= \lambda x. e \end{aligned}$$

Semantics

$$\frac{e_1 \rightarrow e'_1}{e_1 e_2 \rightarrow e'_1 e_2} \quad \frac{e \rightarrow e'}{v e \rightarrow v e'}$$

$$\frac{}{(\lambda x. e) v \rightarrow e\{v/x\}} \beta$$

Rewind: Currying

This is just a function that returns a function:

$$\text{ADD} \triangleq \lambda x. \lambda y. x + y$$

$$\text{ADD } 38 \rightarrow \lambda y. 38 + y$$

$$\text{ADD } 38 \ 4 = (\text{ADD } 38) \ 4 \rightarrow^* 42$$

Informally, you can think of it as a *curried* function that takes two arguments, one after the other.

But that's just a way to get intuition. The λ -calculus only has one-argument functions.

de Bruijn Notation

Another way to avoid the tricky issues with substitution is to use a *nameless* representation of terms.

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Abstractions have lost their variables!

Variables are replaced with numerical indices!

Examples

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$\lambda x.\lambda y.x$	$\lambda.\lambda.1$

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$\lambda x.x$	$\lambda.0$
$\lambda z.z$	$\lambda.0$
$\lambda x.\lambda y.x$	$\lambda.\lambda.1$
$\lambda x.\lambda y.\lambda s.\lambda z.x s (y s z)$	$\lambda.\lambda.\lambda.\lambda.3\ 1\ (2\ 1\ 0)$

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$(\lambda x.x x) (\lambda x.x x)$	$(\lambda.0\ 0) (\lambda.0\ 0)$

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$(\lambda x.\lambda x.x) (\lambda y.y)$	$(\lambda.\lambda.0) (\lambda.0)$

Free variables

To represent a λ -expression that contains free variables in de Bruijn notation, we need a way to map the free variables to integers.

We will work with respect to a map Γ from variables to integers called a *context*.

Examples:

Suppose that Γ maps x to 0 and y to 1.

- Representation of xy is 0 1
- Representation of $\lambda z. xy z = \lambda. 1 2 0$

"SHIFTED" BY +1

Shifting

To define substitution, we will need an operation that shifts by i the variables above a cutoff c :

$$\begin{aligned}\uparrow_c^i(n) &= \begin{cases} n & \text{if } n < c \\ n + i & \text{otherwise} \end{cases} \\ \uparrow_c^i(\lambda.e) &= \lambda.(\uparrow_{c+1}^i e) \\ \uparrow_c^i(e_1 e_2) &= (\uparrow_c^i e_1) (\uparrow_c^i e_2)\end{aligned}$$

The cutoff c keeps track of the variables that were bound in the original expression and so should not be shifted.

The cutoff is 0 initially.

Substitution

Now we can define substitution as follows:

$$\begin{aligned}n\{e/m\} &= \begin{cases} e & \text{if } n = m \\ n & \text{otherwise} \end{cases} \\(\lambda.e_1)\{e/m\} &= \lambda(e_1\{(\uparrow_0^1 e)/m + 1\}) \\(e_1 e_2)\{e/m\} &= (e_1\{e/m\}) (e_2\{e/m\})\end{aligned}$$

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The β rule for terms in de Bruijn notation is just:

$$\frac{}{(\lambda.e_1) e_2 \rightarrow \uparrow_0^{-1} (e_1\{\uparrow_0^1 e_2/0\})} \beta$$

Example

Consider the term $(\lambda u. \lambda v. u x) y$ with respect to a context where $\Gamma(x) = 0$ and $\Gamma(y) = 1$.

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which, in standard notation (with respect to Γ), is the same as $\lambda v. y x$.

Combinators

Another way to avoid the issues having to do with free and bound variable names in the λ -calculus is to work with closed expressions or *combinators*.

With just three combinators, we can encode the entire λ -calculus.

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$$K = \lambda x. \lambda y. x$$

$$S = \lambda x. \lambda y. \lambda z. x z (y z)$$

$$I = \lambda x. x$$

Combinators

We can even define independent evaluation rules that don't depend on the λ -calculus at all.

Behold the “SKI-calculus”:

$$K e_1 e_2 \rightarrow e_1$$

$$S e_1 e_2 e_3 \rightarrow e_1 e_3 (e_2 e_3)$$

$$I e \rightarrow e$$

You would never want to program in this language—it doesn't even have variables!—but it's exactly as powerful as the λ -calculus.

Bracket Abstraction

The function $[x]$ that takes a combinator term M and builds another term that behaves like $\lambda x.M$:

$$\begin{aligned} [x] x &= I \\ [x] N &= K N && \text{where } x \notin fv(N) \\ [x] N_1 N_2 &= S ([x] N_1) ([x] N_2) \end{aligned}$$

The idea is that $([x] M) N \rightarrow M\{N/x\}$ for every term N .

Bracket Abstraction

We then define a function $(e)^*$ that maps a λ -calculus expression to a combinator term:

$$\begin{aligned}(x)^* &= x \\ (e_1 e_2)^* &= (e_1)^* (e_2)^* \\ (\lambda x.e)^* &= [\lambda] (e)^*\end{aligned}$$

Example

As an example, the expression $\lambda x. \lambda y. x$ is translated as follows:

$$\begin{aligned} & (\lambda x. \lambda y. x)^* \\ = & [x] (\lambda y. x)^* \\ = & [x] ([y] x) \\ = & [x] (K x) \\ = & (S ([x] K) ([x] x)) \\ = & S (K K) I \end{aligned}$$

No variables in the final combinator term!

Example

We can check that this behaves the same as our original λ -expression by seeing how it evaluates when applied to arbitrary expressions e_1 and e_2 .

$$\begin{aligned} & (\lambda x. \lambda y. x) e_1 e_2 \\ \rightarrow & (\lambda y. e_1) e_2 \\ \rightarrow & e_1 \end{aligned}$$

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$$\begin{aligned} & (\lambda x. \lambda y. x) e_1 e_2 \\ \rightarrow & (\lambda y. e_1) e_2 \\ \rightarrow & e_1 \end{aligned}$$

and

$$\begin{aligned} & (S (K K) I) e_1 e_2 \\ \rightarrow & (K K e_1) (I e_1) e_2 \\ \rightarrow & K e_1 e_2 \\ \rightarrow & e_1 \end{aligned}$$

SKI Without I

Looking back at our definitions...

$$K e_1 e_2 \rightarrow e_1$$

$$S e_1 e_2 e_3 \rightarrow e_1 e_3 (e_2 e_3)$$

$$I e \rightarrow e$$

...I isn't strictly necessary. It equals S K K.

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...I isn't strictly necessary. It equals S K K.

Our example becomes:

$$S (K K) (S K K)$$

Encodings

The pure λ -calculus contains only functions as values. It is not exactly easy to write large or interesting programs in the pure λ -calculus. We can however encode objects, such as booleans, and integers.

Booleans

We need to define functions TRUE, FALSE, AND, NOT, IF, and other operators that behave as follows:

AND TRUE FALSE \rightarrow FALSE

NOT FALSE \rightarrow TRUE

IF TRUE e_1 e_2 \rightarrow e_1

IF FALSE e_1 e_2 \rightarrow e_2

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NOT FALSE \rightarrow^* TRUE

IF TRUE e_1 e_2 \rightarrow^* e_1

IF FALSE e_1 e_2 \rightarrow^* e_2

Let's start by defining TRUE and FALSE:

TRUE \triangleq $\lambda x. \lambda y. x$

FALSE \triangleq $\lambda x. \lambda y. y$

Booleans

We want the function IF to behave like

$$\lambda b. \lambda t. \lambda f. \text{if } b = \text{TRUE} \text{ then } t \text{ else } f.$$

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$$\text{IF} \triangleq \lambda b. \lambda t. \lambda f. b \ t \ f$$

We can also write the standard Boolean operators.

$$\text{NOT} \triangleq \lambda b. b \ \text{FALSE} \ \text{TRUE}$$

$$\text{AND} \triangleq \lambda b_1. \lambda b_2. b_1 \ b_2 \ \text{FALSE}$$

$$\text{OR} \triangleq \lambda b_1. \lambda b_2. b_1 \ \text{TRUE} \ b_2$$

Church Numerals

Let's encode the natural numbers!

We'll write \bar{n} for the encoding of the number n . The central function we'll need is a *successor* operation:

$$\text{SUCC } \bar{n} = \overline{n + 1}$$

Church Numerals

Church numerals encode a number n as a function that takes f and x , and applies f to x n times.

$$\bar{0} \triangleq \lambda f. \lambda x. x$$

$$\bar{1} \triangleq \lambda f. \lambda x. f x$$

$$\bar{2} \triangleq \lambda f. \lambda x. f(f x)$$

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This makes it easy to write the successor function:

$$\text{SUCC} \triangleq \lambda n. \lambda f. \lambda x. f(n f x)$$

Addition

Given the definition of SUCC, we can define addition. Intuitively, the natural number $n_1 + n_2$ is the result of applying the successor function n_1 times to n_2 .

$$\text{PLUS} \triangleq \lambda n_1. \lambda n_2. n_1 \text{ SUCC } n_2$$