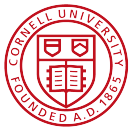
# CS 4110 – Programming Languages and Logics
# Lecture #7: Denotational Semantics




We have now seen two operational models for programming languages: small-step and large-step. In this lecture, we consider a different semantic model, called *denotational semantics*.

The idea in denotational semantics is to express the meaning of a program as the mathematical function that expresses what the program computes. We can think of an IMP program $c$ as a function from stores to stores: given an an initial store, the program produces a final store. For example, the program $\mathsf{foo} := \mathsf{bar} + 1$ can be thought of as a function that when given an input store $\sigma$, produces a final store $\sigma'$ that is identical to $\sigma$ except that it maps foo to the integer $\sigma(\mathsf{bar}) + 1$; that is, $\sigma' = \sigma[\mathsf{foo} \mapsto \sigma(\mathsf{bar}) + 1]$. We will model programs as functions from input stores to output stores. As opposed to operational models, which tell us *how* programs execute, the denotational model shows us *what* programs compute.

## 1 A Denotational Semantics for IMP

For each program $c$, we write $\mathcal{C}[\![c]\!]$ for the *denotation* of $c$, that is, the mathematical function that $c$ represents:

$$\mathcal{C}[\![c]\!] : \mathbf{Store} \rightharpoonup \mathbf{Store}.$$

Note that $\mathcal{C}[\![c]\!]$ is actually a partial function (as opposed to a total function), both because the store may not be defined on the free variables of the program and because program may not terminate for certain input stores. The function $\mathcal{C}[\![c]\!]$ is not defined for non-terminating programs as they have no corresponding output stores.

We will write $\mathcal{C}[\![c]\!]\sigma$ for the result of applying the function $\mathcal{C}[\![c]\!]$ to the store $\sigma$. That is, if $f$ is the function that $\mathcal{C}[\![c]\!]$ denotes, then we write $\mathcal{C}[\![c]\!]\sigma$ to mean the same thing as $f(\sigma)$.

We must also model expressions as functions, this time from stores to the values they represent. We will write $\mathcal{A}[\![a]\!]$ for the denotation of arithmetic expression $a$, and $\mathcal{B}[\![b]\!]$ for the denotation of boolean expression $b$.

$$\mathcal{A}[\![a]\!] : \mathbf{Store} \rightharpoonup \mathbf{Int}$$
$$\mathcal{B}[\![b]\!] : \mathbf{Store} \rightharpoonup \{\mathbf{true}, \mathbf{false}\}$$

Now we want to define these functions. To make it easier to write down these definitions, we will describe (partial) functions using sets of pairs. More precisely, we will represent a partial map $f : A \rightharpoonup B$ as a set of pairs $F = \{(a, b) \mid a \in A \text{ and } b = f(a) \in B\}$ such that, for each $a \in A$, there is at most one pair of the form $(a, \_)$ in the set. Hence $(a, b) \in F$ is the same as $b = f(a)$.

We can now define denotations for IMP. We start with the denotations of expressions:

$$\begin{aligned}
\mathcal{A}[\![n]\!] &= \{(\sigma, n)\} \\
\mathcal{A}[\![x]\!] &= \{(\sigma, \sigma(x))\} \\
\mathcal{A}[\![a_1 + a_2]\!] &= \{(\sigma, n) \mid (\sigma, n_1) \in \mathcal{A}[\![a_1]\!] \wedge (\sigma, n_2) \in \mathcal{A}[\![a_2]\!] \wedge n = n_1 + n_2\}
\end{aligned}$$

$$\begin{aligned}
\mathcal{B}[\![\textbf{true}]\!] &= \{(\sigma, \textbf{true})\} \\
\mathcal{B}[\![\textbf{false}]\!] &= \{(\sigma, \textbf{false})\} \\
\mathcal{B}[\![a_1 < a_2]\!] &= \{(\sigma, \textbf{true}) \mid (\sigma, n_1) \in \mathcal{A}[\![a_1]\!] \wedge (\sigma, n_2) \in \mathcal{A}[\![a_2]\!] \wedge n_1 < n_2\} \cup \\
&\quad \{(\sigma, \textbf{false}) \mid (\sigma, n_1) \in \mathcal{A}[\![a_1]\!] \wedge (\sigma, n_2) \in \mathcal{A}[\![a_2]\!] \wedge n_1 \geq n_2\}
\end{aligned}$$

The denotations for commands are as follows:

$$\begin{aligned}
\mathcal{C}[\![\textbf{skip}]\!] &= \{(\sigma, \sigma)\} \\
\mathcal{C}[\![x := a]\!] &= \{(\sigma, \sigma[x \mapsto n]) \mid (\sigma, n) \in \mathcal{A}[\![a]\!]\} \\
\mathcal{C}[\![c_1; c_2]\!] &= \{(\sigma, \sigma') \mid \exists \sigma''.\ ((\sigma, \sigma'') \in \mathcal{C}[\![c_1]\!] \wedge (\sigma'', \sigma') \in \mathcal{C}[\![c_2]\!])\}
\end{aligned}$$

Note that $\mathcal{C}[\![c_1; c_2]\!] = \mathcal{C}[\![c_2]\!] \circ \mathcal{C}[\![c_1]\!]$, where $\circ$ is the composition of relations, defined as follows: if $R_1 \subseteq A \times B$ and $R_2 \subseteq B \times C$ then $R_2 \circ R_1 \subseteq A \times C$ is $R_2 \circ R_1 = \{(a, c) \mid \exists b \in B.\ (a, b) \in R_1 \wedge (b, c) \in R_2\}$.) If $\mathcal{C}[\![c_1]\!]$ and $\mathcal{C}[\![c_2]\!]$ are total functions, then $\circ$ is function composition.

$$\begin{aligned}
\mathcal{C}[\![\textbf{if } b \textbf{ then } c_1 \textbf{ else } c_2]\!] &= \{(\sigma, \sigma') \mid (\sigma, \textbf{true}) \in \mathcal{B}[\![b]\!] \wedge (\sigma, \sigma') \in \mathcal{C}[\![c_1]\!]\} \cup \\
&\quad \{(\sigma, \sigma') \mid (\sigma, \textbf{false}) \in \mathcal{B}[\![b]\!] \wedge (\sigma, \sigma') \in \mathcal{C}[\![c_2]\!]\} \\
\mathcal{C}[\![\textbf{while } b \textbf{ do } c]\!] &= \{(\sigma, \sigma) \mid (\sigma, \textbf{false}) \in \mathcal{B}[\![b]\!]\} \cup \\
&\quad \{(\sigma, \sigma') \mid (\sigma, \textbf{true}) \in \mathcal{B}[\![b]\!] \wedge \exists \sigma''.\ ((\sigma, \sigma'') \in \mathcal{C}[\![c]\!] \wedge (\sigma'', \sigma') \in \mathcal{C}[\![\textbf{while } b \textbf{ do } c]\!])\}
\end{aligned}$$

But now we have a problem: the last "definition" is not really a definition because it expresses $\mathcal{C}[\![\textbf{while } b \textbf{ do } c]\!]$ in terms of itself! This is not a definition but a recursive equation. What we want is the solution to this equation.

## 2 Fixed points

We gave a recursive equation that the function $\mathcal{C}[\![\textbf{while } b \textbf{ do } c]\!]$ must satisfy. To understand some of the issues involved, let's consider a simpler example. Consider the following equation for a function $f : \mathbb{N} \to \mathbb{N}$.

$$f(x) = \begin{cases} 0 & \text{if } x = 0 \\ f(x-1) + 2x - 1 & \text{otherwise} \end{cases} \tag{1}$$

This is not a definition for $f$, but rather an equation that we want $f$ to satisfy. What function, or functions, satisfy this equation for $f$? The only solution to this equation is the function $f(x) = x^2$.

In general, there may be no solutions for a recursive equation (e.g., there are no functions $g : \mathbb{N} \to \mathbb{N}$ that satisfy the recursive equation $g(x) = g(x) + 1$), or multiple solutions (e.g., find two functions $g : \mathbb{R} \to \mathbb{R}$ that satisfy $g(x) = 4 \times g(\frac{x}{2})$).

We can compute solutions to such equations by building successive approximations. Each approximation is closer and closer to the solution. To solve the recursive equation for $f$, we start with the partial function $f_0 = \emptyset$ (i.e., $f_0$ is the empty relation; it is a partial function with the empty set for it's domain). We compute successive approximations using the recursive equation.

$$
\begin{aligned}
f_0 &= \emptyset \\
f_1 &= \begin{cases} 0 & \text{if } x = 0 \\ f_0(x-1) + 2x - 1 & \text{otherwise} \end{cases} \\
&= \{(0,0)\} \\
f_2 &= \begin{cases} 0 & \text{if } x = 0 \\ f_1(x-1) + 2x - 1 & \text{otherwise} \end{cases} \\
&= \{(0,0),(1,1)\} \\
f_3 &= \begin{cases} 0 & \text{if } x = 0 \\ f_2(x-1) + 2x - 1 & \text{otherwise} \end{cases} \\
&= \{(0,0),(1,1),(2,4)\} \\
&\vdots
\end{aligned}
$$

This sequence of successive approximations $f_i$ gradually builds the function $f(x) = x^2$.

We can model this process of successive approximations using a higher-order function $F$ that takes one approximation $f_k$ and returns the next approximation $f_{k+1}$:

$$F : (\mathbb{N} \rightharpoonup \mathbb{N}) \to (\mathbb{N} \rightharpoonup \mathbb{N})$$

where

$$(F(f))(x) = \begin{cases} 0 & \text{if } x = 0 \\ f(x-1) + 2x - 1 & \text{otherwise} \end{cases}$$

A solution to the recursive equation 1 is a function $f$ such that $f = F(f)$. In general, given a function $F : A \to A$, we have that $a \in A$ is a *fixed point* of $F$ if $F(a) = a$. We also write $a = \text{fix}(F)$ to indicate that $a$ is a fixed point of $F$.

So the solution to the recursive equation 1 is a fixed-point of the higher-order function $F$. We can compute this fixed point iteratively, starting with $f_0 = \emptyset$ and at each iteration computing $f_{k+1} = F(f_k)$. The fixed point is the limit of this process:

$$
\begin{aligned}
f &= \text{fix}(F) \\
&= f_0 \cup f_1 \cup f_2 \cup f_3 \cup \ldots \\
&= \emptyset \cup F(\emptyset) \cup F(F(\emptyset)) \cup F(F(F(\emptyset))) \cup \ldots \\
&= \bigcup_{i \geq 0} F^i(\emptyset)
\end{aligned}
$$