CS 3110

Efficiency

Prof. Clarkson
Fall 2015

Today's music: Opening theme from *The Big O*
(THE ビッグオ)
by Toshihiko Sahashi

Review

Previously in 3110:

- Reasoning about correctness of programs

Today:

- Reasoning about efficiency of programs

Question

Which of the following would you prefer?

A. O(n^2)
B. O(log(n))
C. O(n)
D. They're all good
E. I thought this was 3110, not Algo

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B. **O(log(n))**
C. O(n)
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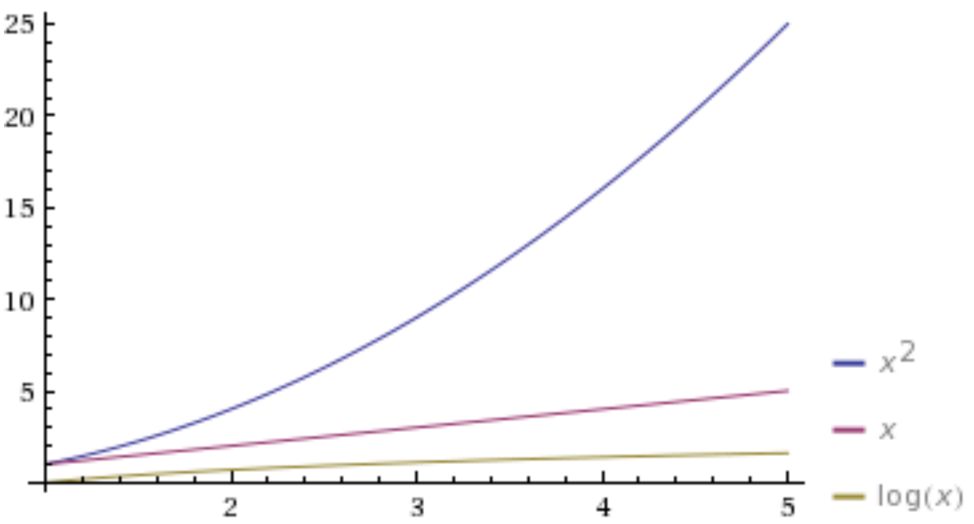


Performance

- You've built beautiful, elegant, functional code
- You've organized it into modules with clear specifications
- You've ascertained the correctness of your code through testing or even formal verification

- **Now**, you begin to worry about performance
  - Some part of code is too slow
  - You want to understand the *efficiency* of a data structure
  - You want to find a more *efficient* algorithm

What is "efficiency"?

Attempt #1: An algorithm is efficient if, when implemented, it runs quickly on particular input instances

...problems with that?

What is "efficiency"?

Attempt #1: An algorithm is efficient if, when implemented, it runs quickly on particular input instances

Incomplete list of problems:

- Inefficient algorithms can run quickly on small test cases
- Fast processors and optimizing compilers can make inefficient algorithms run quickly
- Efficient algorithms can run slowly when coded sloppily
- Some input instances are harder than others
- Efficiency on small inputs doesn't imply efficiency on large inputs
- Some clients can afford to be more patient than others; quick for me might be slow for you

Lessons learned from attempt #1

Lesson 1: Time as measured by a clock is not the right metric

- Want a metric that is reasonably independent of hardware, compiler, other software running, etc.
- **idea:** number of steps taken (by dynamic semantics) during evaluation of program
  - steps are independent of implementation details
  - But: each step might really take a different amount of time?
    - creating a closure, looking up a variable, computing an addition
  - in practice, the difference isn't really big enough to matter

Lessons learned from attempt #1

Lesson 2: Running time on particular input instances is not the right metric

- Want a metric that can predict running time on **any** input instance
- **idea:** size of the input instance
  - make metric be a function of input size
  - (combined with lesson 1) specifically, the maximum number of steps for an input of that size
  - But: particular inputs of the same size might really take a different amount of time?
    - multiplying arbitrary matrices vs. multiplying by all zeros
  - in practice, size matters more

Lessons learned from attempt #1

Lesson 3: Quickness is not the right metric

- Want a metric that is reasonably objective; independent of subjective notions of what is fast
- **idea:** beats brute-force search
  - *brute force:* enumerate all the answers one by one, check and see whether the answer is right
    - the simple, dumb solution to nearly any algorithmic problem
    - related idea: guess an answer, check whether correct
      e.g., bogosort
  - but *by how much* is enough to beat brute-force search?

Lessons learned from attempt #1

Lesson 3: Quickness is not the right metric

- **better idea:** polynomial time
  - (combined with ideas from previous two lessons)
    can express maximum number of steps as a polynomial function of the size N of input, e.g.,
    - aN^2 + bN + c
  - But: some polynomials might be too big to be quick (N^100)?
  - But: some non-polynomials might be quick enough (N^(1+.02*(log N)))?
  - in practice, polynomial time really does work

What is "efficiency"?

Attempt #2: An algorithm is efficient if its maximum number of steps of execution is polynomial in the size of its input.

let's give that a try...

Analysis of running time

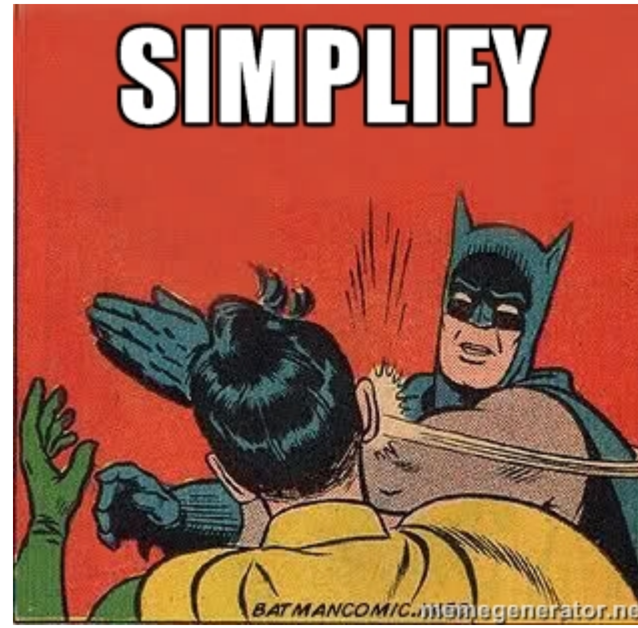
INSERTION-SORT(A)	*cost*	*times*
for j = 2 **to** A.length	c_1	n
key = *A*[*j*]	c_2	$n - 1$
// Insert *A*[*j*] into the sorted sequence *A*[1 .. *j* - 1]	0	$n - 1$
i = *j* - 1	c_4	$n - 1$
while i > 0 and *A*[*i*] < *key*	c_5	$\sum_{j=2}^{n} t_j$
A[*i* + 1] = *A*[*i*]	c_6	$\sum_{j=2}^{n} (t_j - 1)$
i = *i* - 1	c_7	$\sum_{j=2}^{n} (t_j - 1)$
A[*i* + 1] = *key*	c_8	$n - 1$

[Cormen et al. *Introduction to Algorithms*, 3rd ed, 2009]

Analysis of running time

```
INSERTION-SORT(A)
for j = 2 to A.length
    key = A[j]
    // Insert A[j] into the sorted
        sequence A[1 .. j - 1]
    i = j - 1
    while i > 0 and A[i] < key
        A[i + 1] = A[i]
        i = i - 1
    A[i + 1] = key
```

cost	*times*
c_1	n
c_2	$n - 1$
0	$n - 1$
c_4	$n - 1$
c_5	$\sum_{j=2}^{n} t_j$
c_6	$\sum_{j=2}^{n} (t_j - 1)$
c_7	$\sum_{j=2}^{n} (t_j - 1)$
c_8	$n - 1$



The running time of the algorithm is the sum of running times for each statement executed; a statement that takes c_i steps to execute and executes n times will contribute $c_i n$ to the total running time.[6] To compute $T(n)$, the running time of INSERTION-SORT on an input of n values, we sum the products of the *cost* and *times* columns, obtaining

$$T(n) = c_1 n + c_2(n-1) + c_4(n-1) + c_5 \sum_{j=2}^{n} t_j + c_6 \sum_{j=2}^{n} (t_j - 1) + c_7 \sum_{j=2}^{n} (t_j - 1) + c_8(n-1) .$$

[Cormen et al. *Introduction to Algorithms*, 3rd ed, 2009]

Precision of running time

- Precise bounds are exhausting to find
- Precise bounds are to some extent meaningless
  - Are those constants c1..c8 really useful?
  - If it takes 25 steps in high level language, but compiled down to assembly would take 10x more steps, is the precision useful?
  - Caveat: if you're building code that flies an airplane or controls a nuclear reactor, you do care about precise, real-time guarantees

Some simplified running times

max # steps as function of N

size of input

	N	N^2	N^3	2^N
N=10	< 1 sec	< 1 sec	< 1 sec	< 1 sec
N=100	< 1 sec	< 1 sec	1 sec	10^17 years
N=1,000	< 1 sec	1 sec	18 min	very long
N=10,000	< 1 sec	2 min	12 days	very long
N=100,000	< 1 sec	3 hours	32 years	very long
N=1,000,000	1 sec	12 days	10^4 years	very long

assuming 1 microsecond/step

very long = more years than the estimated number of atoms in universe

Simplifying running times

- Rather than 1.62N^2 + 3.5N + 8 steps, we would rather say that running time "grows like N^2"
  - identify broad classes of algorithm with similar performance
- Ignore the *low-order terms*
  - e.g., ignore 3.5N+8
  - Why? For big N, N^2 is much, much bigger than N
- Ignore the *constant factor* of high-order term
  - e.g., ignore 1.62
  - Why? For classifying algorithms, constants aren't meaningful
    - Code run on my machine might be a constant factor faster or slower than on your machine, but that's not a property of the algorithm
  - Caveat: Performance tuning real-world code actually can be about getting the constants to be small!
- Abstraction to an imprecise quantity

Imprecise abstractions

- OCaml's **`int`** type is an abstraction of a subset of Z
  - don't know which int when reasoning about the type of an expression
- ±1 is an abstraction of {1,-1}
  - don't know which when manipulating it in a formula
- Here's a new one: Big Ell
  - L(e) represents a natural number whose value is less than or equal to e
  - precisely, L(e) = {m | 0 <= m <= e}
  - e.g., L(5) = {0, 1, 2, 3, 4, 5}

Manipulating Big Ell

- What is 1 + L(5)?
- Trick question!
  - Replace L(5) with set: $1 + \{0..5\}$
  - But + is defined on ints, not sets of ints
- We could distribute the + over the set: $\{1+0, ..., 1+5\} = \{1..6\}$
  - That is, a set of values, one for each possible instantiation of L(5)
- Note that $\{1..6\} \subseteq \{0..6\} = L(6)$
- So we could say that $1 + L(5) \subseteq L(6)$

Question #2

What is L(2) + L(3)?

Hint: set of values, one for each possible instantiation of L(2) and of L(3)

A. L(2) + L(3) $\subseteq$ L(2)

B. L(2) + L(3) $\subseteq$ L(3)

C. L(2) + L(3) $\subseteq$ L(4)

D. L(2) + L(3) $\subseteq$ L(5)

E. L(2) + L(3) $\subseteq$ L(6)

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Question #3

What is $L(2) * L(3)$?

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D. L(2) * L(3) ⊆ L(5)
E. L(2) * L(3) ⊆ L(6)

A little trickier...

What is $2^{L(3)}$?

- $L(3) = \{0..3\}$
- So $2^{L(3)}$ could be any of $\{2^0, \ldots, 2^3\} = \{1, 2, 4, 8\}$
- And $\{1,2,4,8\} \subseteq L(8) = L(2^3)$
- Therefore $2^{L(3)} \subseteq L(2^3)$

...we can use this idea of Big Ell to invent an imprecise abstraction for running times

Big Oh, take 1

- Recall: we're interested in running time as a function of input size
- Recall: L(e) represents any natural number that is less than or equal to a natural number e
- "New" imprecise abstraction: Big Oh
  - O(g) represents any function that is less than or equal to function g, for every input n.
  - Big Oh is a higher-order version of Big Ell: generalize from naturals to functions on naturals
  - precisely, O(g) = {f | forall n, f(n) <= g(n)}
  - e.g., O(fun n -> 2n) = {f | forall n, f(n) <= 2n}
    - (fun n -> n) ∈ O(fun n -> 2n)
    - note: that's a mathematical function written in OCaml notation, not an OCaml function; that's why I'm not putting it in `typewriter font`
- For simplicity, let's assume function inputs and outputs are non-negative (since input size and running time won't be negative)

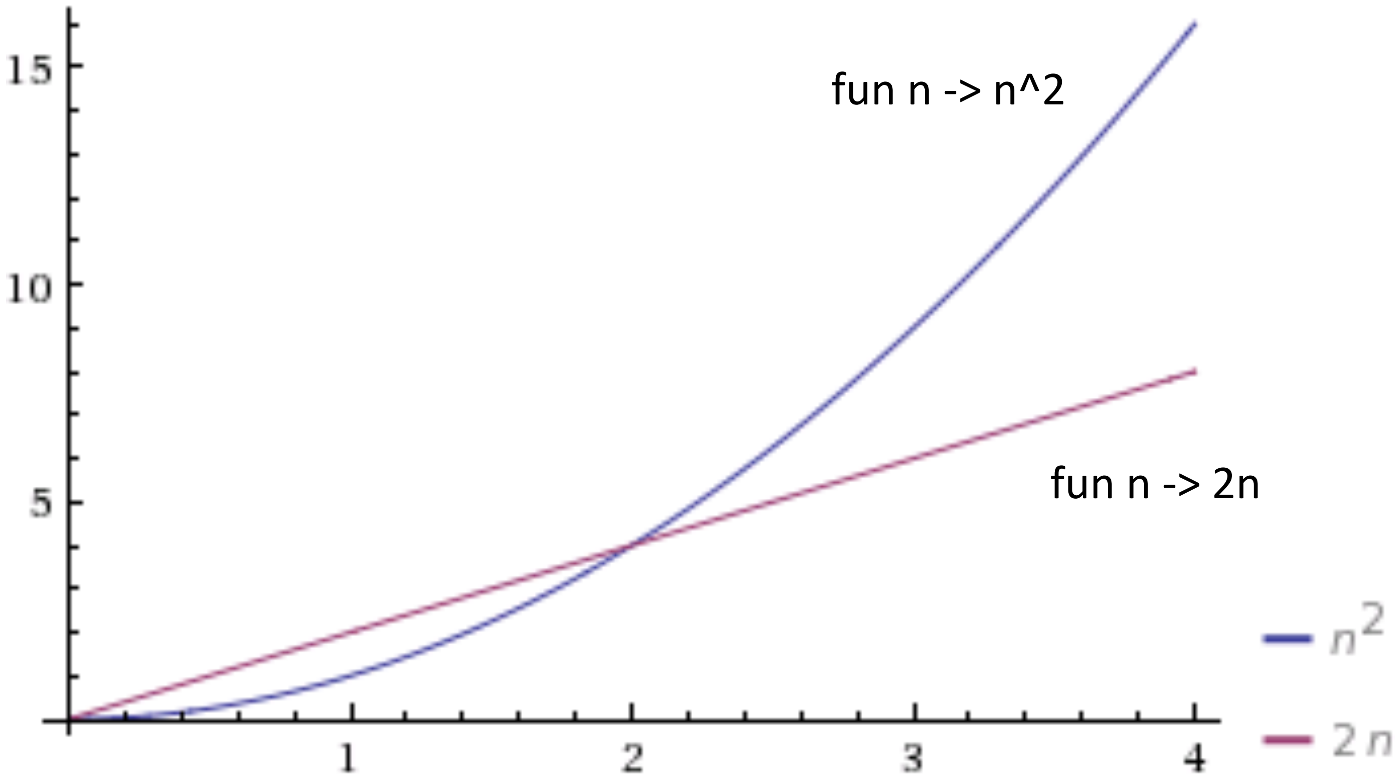
Big Oh, take 2

Recall: we want to ignore constant factors

- O(g) represents any function that is less than or equal to function g times some positive constant c, for every input n.
- precisely, O(g) = {f | exists c>0, forall n,
  f(n) <= c * g(n)}
- e.g., O(fun n -> n^3) = {f | exists c>0, forall n,
  f(n) <= c * n^3}
  - (fun n -> 3*n^3) ∈ O(fun n -> n^3)
    because 3*n^3 <= c * n^3, where c = 3 (or c=4, ...)

Big Oh, take 3

Recall: we care about what happens at scale



could just build a lookup table for inputs in the range 0..2

Big Oh, take 3

Recall: we care about what happens at scale

- O(g) represents any function that is less than or equal to function g times some positive constant c, for every input n greater than or equal to some positive constant n0.
- precisely, O(g) = {f | exists c>0, n0>0, forall n >= n0,
  f(n) <= c * g(n)}
- e.g., O(fun n -> n^2) = {f | exists c>0, n0>0, forall n >= n0,
  f(n) <= c * n^2}
  - (fun n -> 2n) ∈ O(fun n -> n^2)
    because 2n <= c * n^2, where c = 2, for all n >= 1

Big Oh

The important, final definition you should know:

$$
\begin{aligned}
O(g) = \{f \mid & \text{ exists } c>0,\ n0>0, \\
& \text{forall } n >= n0, \\
& f(n) <= c * g(n)\}
\end{aligned}
$$

Big Oh Notation: Warning 1

Instead of

$O(g) = \{f \mid ...$

most authors write

$O(g(n)) = \{f(n) \mid ...$

- They don't really mean g applied to n; they mean a function g parameterized on input n but not yet applied
- Maybe they never studied functional programming ☺

Big Oh Notation: Warning 2

Instead of

(fun n -> 2n) ∈ O(fun n -> n^2)

all authors write

2n = O(n^2)

- Your instructor has always found this abusage distressing
- Yet henceforth he will follow the convention ☺
  - The standard defense is that = should be read here as "is" not as "equals"
  - Be careful: one-directional equality!

A Theory of Big Oh

- reflexivity: $f = O(f)$
- *(no symmetry condition for Big Oh; there is one for* Big Theta)
- transitivity: $f = O(g)$ /\ $g = O(h)$ => $f = O(h)$
- $c * O(f) = O(f)$
- $O(c * f) = O(f)$
- $O(f) * O(g) = O(f * g)$
  - where f * g means (fun n -> f(n)*g(n))
- …

Useful to know these equalities so that you don't have to keep re-deriving them from first principles

What is "efficiency"?

Final attempt: An algorithm is efficient if its worst-case running time is O(N^d) for some constant d and for input size N.

Running times of some algorithms

- **O(1): constant:** access an element of an array (of length n)
- **O(log n): logarithmic:** binary search through sorted array of length n
- **O(n): linear:** maximum element of list of length n
- **O(n log n): linearithmic:** mergesort a list of length n
- **O(n^2): quadratic:** bubblesort an array of length n
- **O(n^3): cubic:** matrix multiplication of n-by-n matrices
- **O(2^n): exponential:** enumerate all integers of bit length n

...some of these are not obvious, require proof

Upcoming events

- [today] A5 due, including Async and design phase of project
- [in next week] Design review meetings
- [next Thursday] Prelim 2

This is efficient.

THIS IS 3110