CS 3110

Behavioral Equivalence

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Fall 2015

Today's music: *Soul Bossa Nova* by Quincy Jones

Review

Previously in 3110:

- Functional programming
- Modular programming
- Interpreters
- Imperative and concurrent programming

Today:

- Reasoning about correctness of programs

Building Reliable Software

- Suppose you work at (or run) a software company.

- Suppose you've sunk 30+ person-years into developing the "next big thing":
  - Boeing Dreamliner2 flight controller
  - Autonomous vehicle control software for Nissan
  - Gene therapy DNA tailoring algorithms
  - Super-efficient green-energy power grid controller

- How do you avoid disasters?
  - Turns out software endangers lives
  - Turns out to be impossible to build software

Approaches to Reliability

- Social
  - Code reviews
  - Extreme/Pair programming
- Methodological
  - Design patterns
  - Test-driven development
  - Version control
  - Bug tracking
- Technological
  - Static analysis
    (“lint” tools, FindBugs, …)
  - Fuzzers
- Mathematical
  - Sound type systems
  - “Formal” verification

Less formal: Techniques may miss problems in programs

All of these methods should be used!

Even the most formal can still have holes:

- did you prove the right thing?
- do your assumptions match reality?

More formal: eliminate *with certainty* as many problems as possible.

Testing vs. Verification

Testing:

- Cost effective
- Guarantee that program is correct on tested inputs and in tested environments

Verification:

- Expensive
- Guarantee that program is correct on all inputs and in all environments

Edsger W. Dijkstra

(1930-2002)

Turing Award Winner (1972)

For eloquent insistence and practical demonstration that programs should be composed correctly, not just debugged into correctness

"Program testing can at best show the presence of errors but never their absence."

Verification

- In the 1970s, scaled to about tens of LOC
- Now, research projects scale to real software:
  - CompCert: verified C compiler
  - seL4: verified microkernel OS
  - Ynot: verified DBMS, web services
- In another 40 years?

Our trajectory

- Proofs about functions
- Proofs about variants
- Proofs about modules

- We're not trying to get all the way to fully machine-checked correctness proofs of large programs
- Rather:
  - help you understand what it means to be correct
  - help you organize your thoughts about correctness of code you write

- Important caveat: no side-effects!
  - specifically, no mutability or I/O
  - exceptions will be fine

Example

```
let rec even n =
  match n with
    | 0 -> true
    | 1 -> false
    | n -> even (n-2)
```

Theorem. For all natural numbers `n`, it holds that `even (2*n)` is `true`.

Example

```
(* precondition:  n >= 0 *)
(* postcondition: (fact n) = n! *)
let rec fact n =
  if n=0 then 1
  else n * fact (n-1)
```

Theorem. `fact` is *correct*—it satisfies its specification.

Example

```
let rec length = function
   | [] -> 0
   | _::xs -> 1 + length xs

let rec append xs1 xs2 = match xs1 with
   | [] -> xs2
   | h::t -> h :: append t xs2
```

Theorem. For all lists `xs` and `ys`, it holds that `length (append xs ys)` is `length xs + length ys`.

EQUIVALENCE OF EXPRESSIONS

Behavioral equivalence

- **Behavioral equivalence:** two expressions behave the same
  - always evaluate to same value

Question

Which of these expressions is behaviorally equivalent to `42`?

A. `if b then 42 else 42`
(for an arbitrary Boolean expression b)

B. `let _ = f x in 42`
(for an arbitrary function f and argument x)

C. `List.hd [42]`

D. All of the above

E. None of the above

Question

Which of these expressions is behaviorally equivalent to `42`?

A. `if b then 42 else 42`
   (for an arbitrary Boolean expression b)

B. `let _ = f x in 42`
   (for an arbitrary function f and argument x)

C. `List.hd [42]`

D. All of the above

E. None of the above

Behavioral equivalence

- **Behavioral equivalence:** two expressions behave the same
  - always evaluate to same value
  - (or always raise the same exception)
  - (or always *diverge*: don't terminate)
- Write as `e1~e2`
  - I would much prefer `e1≡e2`, but that symbol isn't available in plain text

Behavioral equivalence

Fundamental *axioms* about when expressions are behaviorally equivalent:

- **eval:** if `e1-->*e2` then `e1~e2`
- **alpha:** if `e1` differs from `e2` only by consistent renaming of variables then `e1~e2`
- **sugar:** if `e1` is syntactic sugar for `e2` then `e1~e2`

Behavioral equivalence

Facts (theorems) about behavioral equivalence:

- **reflexive:** `e ~ e`
- **symmetric:** if `e1 ~ e2` then `e2 ~ e1`
- **transitive:** if `e1 ~ e2` and `e2 ~ e3` then `e1 ~ e3`

...that is, ~ is an *equivalence relation*

Easy example with ~

```
let easy x y z = x * (y + z)

Theorem:  easy 1 20 30 ~ 50
Proof:
  easy 1 20 30
~ 50                  (by eval)
QED
```

Another easy example

```
let easy x y z = x * (y + z)

Theorem:
  for all ints n and m, easy 1 n m ~ n + m

Proof:
  easy 1 n m
~ n + m              (by eval)
QED
```

Not so!

Evaluation with unknown values

- That proof wasn't valid according to the small-step semantics:
  - `easy 1 n m -/>`
  - because `n` and `m` aren't strictly speaking values
  - they might as well be, though...
- *Symbolic values:* they stand for a value
  - Think of them as "mathematical variables" as opposed to "program variables"
  - They are values; we just don't know what they are
  - We'll allow the semantics to consider them as values
- So we can allow evaluation to continue:
  - `easy 1 n m -> x*(y+z){1/x}{n/y}{m/z} -> 1*(n+m) -/>`
  - because `n+m` isn't strictly speaking a value
  - it might as well be, though; guaranteed to produce a value at runtime...

Valuable expressions

- *Valuable*: guaranteed to produce a value
  - No exceptions
  - Always terminates
- If an expression is valuable, then we may use it as though it were already a value in the semantics
- So we can allow evaluation to continue:

```
easy 1 n m
-> x*(y+z){1/x}{n/y}{m/z}
-> 1*(n+m)
-> n+m
```

Valuable expressions

Definition of *valuable:*

- a (symbolic) value is valuable
- a variable is valuable
  - at run-time, will be replaced by a value
- any pair, record, or variant built out of valuable expressions is valuable
- an `if` expression is valuable if all its subexpressions are valuable
- a pattern-matching expression is valuable if it is exhaustive
  - non-exhaustive could raise exception at run time
- a function application is valuable if the argument is valuable and the function is *total*: guaranteed to terminate with a value
  - + is total
  - / is *partial,* as is `List.hd`

Why we need totality

```
let rec forever x = forever ()
let one x = 1
```

If we didn't require functions to be total, we would conclude

```
  one (forever ())
  -> 1{forever()/x} = 1
```

hence

```
  one (forever ()) ~ 1
```

which violates the definition of behavioral equivalence

Why we need totality

```
let one x = 1
```

If we didn't require functions to be total, we would conclude

```
one (List.hd [])
-> 1{List.hd []/x} = 1
```

hence

```
one (List.hd []) ~ 1
```

which violates the definition of behavioral equivalence

Using valuable expressions

```
let easy x y z = x * (y + z)

Theorem:  for all ints a, b, and c,
  easy a b c ~ easy a c b

Proof:
  easy a b c
~ a * (b + c)    (by eval)
~ a * (c + b)    (???)
~ easy a c b     (by eval, symm.)
QED
```

"By math"

Assume basic algebraic properties of the OCaml built-in operators:

- `(r+s)+t ~ r + (s + t)`
- `r+s ~ s+r`
- `r+0 ~ 0+r ~ r`
- `r + (-r) ~ 0`
- `r*s ~ s*r`
- `(r*s)*t ~ r*(s*t)`
- `r*0 ~ 0*r ~ r`
- `r*1 ~ 1*r ~ r`
- `r*(s+t) ~ (r*s)+(r*t)`
- `(r+s)*t ~ (r*t)+(s*t)`
- etc.

where `r,s,t` must (in general) be valuable

"By math"

Allow use of other mathematical operators that aren't built-in to OCaml:

- Integer exponentiation
- Factorial
- etc.

All arguments must be valuable

e.g.

```
(k+1)! ~ (k+1)*(k!) (by math)
```

Using valuable expressions

```
let easy x y z = x * (y + z)

Theorem:  for all ints a, b, and c,
easy a b c ~ easy a c b

Proof:
   easy a b c
~ a * (b + c)    (by eval)
~ a * (c + b)    (by math)
~ easy a c b     (by eval, symm.)
QED
```

Doubles are even

```
(* requires:  n >= 0 *)
let rec even n =
  match n with
    | 0 -> true
    | 1 -> false
    | n -> even (n-2)
```

Naturals: integers >= 0
We ignore the limits of
machine arithmetic here.

```
Theorem:
for all natural numbers n,
  even (2*n) ~ true.
```

Doubles are even

```
let rec even n =
  match n with
    | 0 -> true
    | 1 -> false
    | n -> even (n-2)
```

```
Theorem:
for all natural numbers n, even (2*n) ~ true.

Proof: by induction. QED
```

A PL theorist's favorite proof. :)

Doubles are even

```
let rec even n =
  match n with
    | 0 -> true
    | 1 -> false
    | n -> even (n-2)
```

```
Theorem:
for all natural numbers n, even (2*n) ~ true.

Proof: by induction on n

Case:  n is 0
Show:  even (2*0) ~ true

  even (2*0)
~ true        (eval)
```

Doubles are even

```
let rec even n =
  match n with
    | 0 -> true
    | 1 -> false
    | n -> even (n-2)
```

```
Theorem:
for all natural numbers n, even (2*n) ~ true.

Proof: by induction on n

Case:  n is k+1, where k >= 0
IH:    even (2*k) ~ true
Show:  even (2*(k+1)) ~ true

  even (2*(k+1))
~ even (2*k+2)      (???)
```

Question

What would justify this proof step?

```
even (2*(k+1)) ~ even (2*k+2)
```

A. math
B. eval
C. transitivity
D. All the above together
E. None of the above

Question

What would justify this proof step?

```
even (2*(k+1)) ~ even (2*k+2)
```

A. math
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Congruence

A deep fact about behavioral equivalence:

congruence:
if `e1 ~ e2` then `e{e1/x} ~ e{e2/x}`

aka *substitution of equals for equals* and *Leibniz equality*

Congruence is hugely important: enables local reasoning

- replace small part of large program with an equivalent small part
- conclude equivalence of large programs without having to do large proof!

Doubles are even

```
let rec even n =
  match n with
    | 0 -> true
    | 1 -> false
    | n -> even (n-2)
```

```
Theorem:
for all natural numbers n,

Proof: by induction on n

Case:  n is k+1, where k >
IH:    even (2*k) ~ true
Show:  even (2*(k+1)) ~ tr

  even (2*(k+1))
~ even (2*k+2)        (math,congr.)
~ even (2*k+2-2)      (eval,k>=0)
~ even (2*k)          (math,congr.)
~ true                (IH)

QED
```

math shows

```
2*(k+1)~2*k+2
```

congruence shows

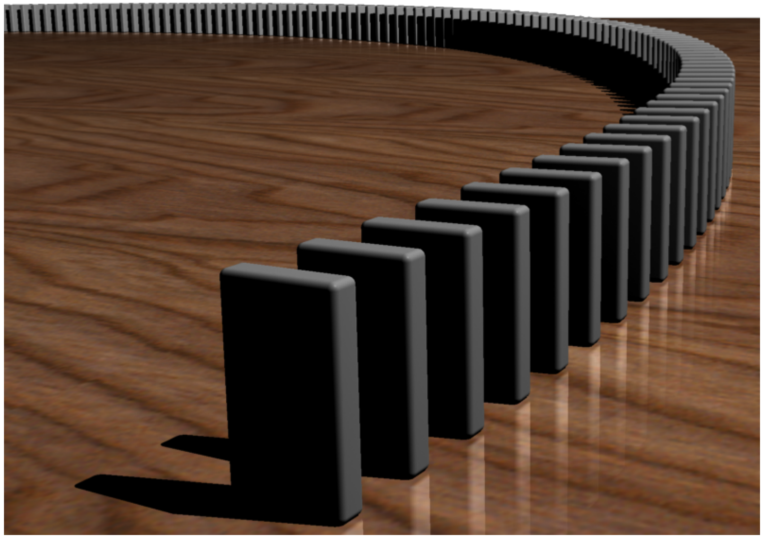
```
  (even x){2*(k+1)/x}
~ (even x){2*k+2   /x}
```

Review: Induction on natural numbers

```
Theorem:
for all natural numbers n, P(n).

Proof: by induction on n

Case:  n is 0
Show:  P(0)

Case:  n is k+1
IH:    P(k)
Show:  P(k+1)

QED
```

Induction principle

```
for all properties P of natural numbers,
  if P 0
  and (for all n, P n implies P (n+1))
  then (for all n, P n)
```



Upcoming events

- [soon] A5 out; includes design milestone of project which you can start immediately

This is well behaved.

THIS IS 3110

Acknowledgements

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Those materials are in turn based on materials from Prof. Bob Harper (Cornell PhD 1985) et al. at Carnegie Mellon University from the course 15-150 Functional Programming.

Academic genealogy: Constable -> Harper -> Morrisett -> Walker (-> means *advised*)