Adequacy and Complete Axiomatization for Timed Modal Logic

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Joint work with

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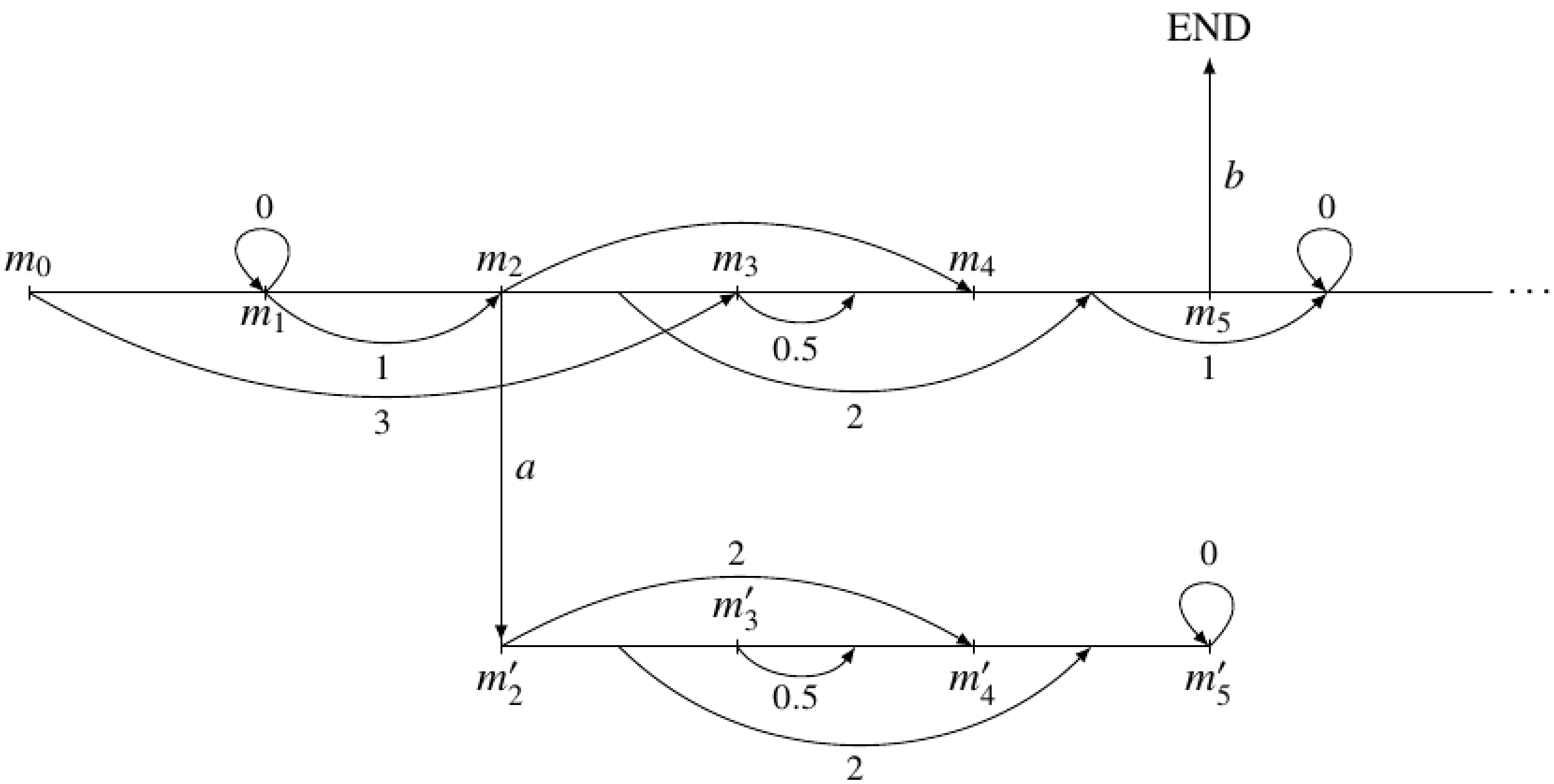
Motivation

- To model real-time behaviors:
  - Timed transition system; timed automata
  - $MTL, MITL, TPTL, ECL$ –- extensions of LTL
  - $TCTL, T_\mu, L_\nu$ -- extensions of CTL, μ-calculus
- Our achievements:
  - Adequacy of TML (Timed Modal Logic)
  - Undecidability of satisfiability problem for TML
  - Strong-complete axiomatization

Timed Transition System (TTS)



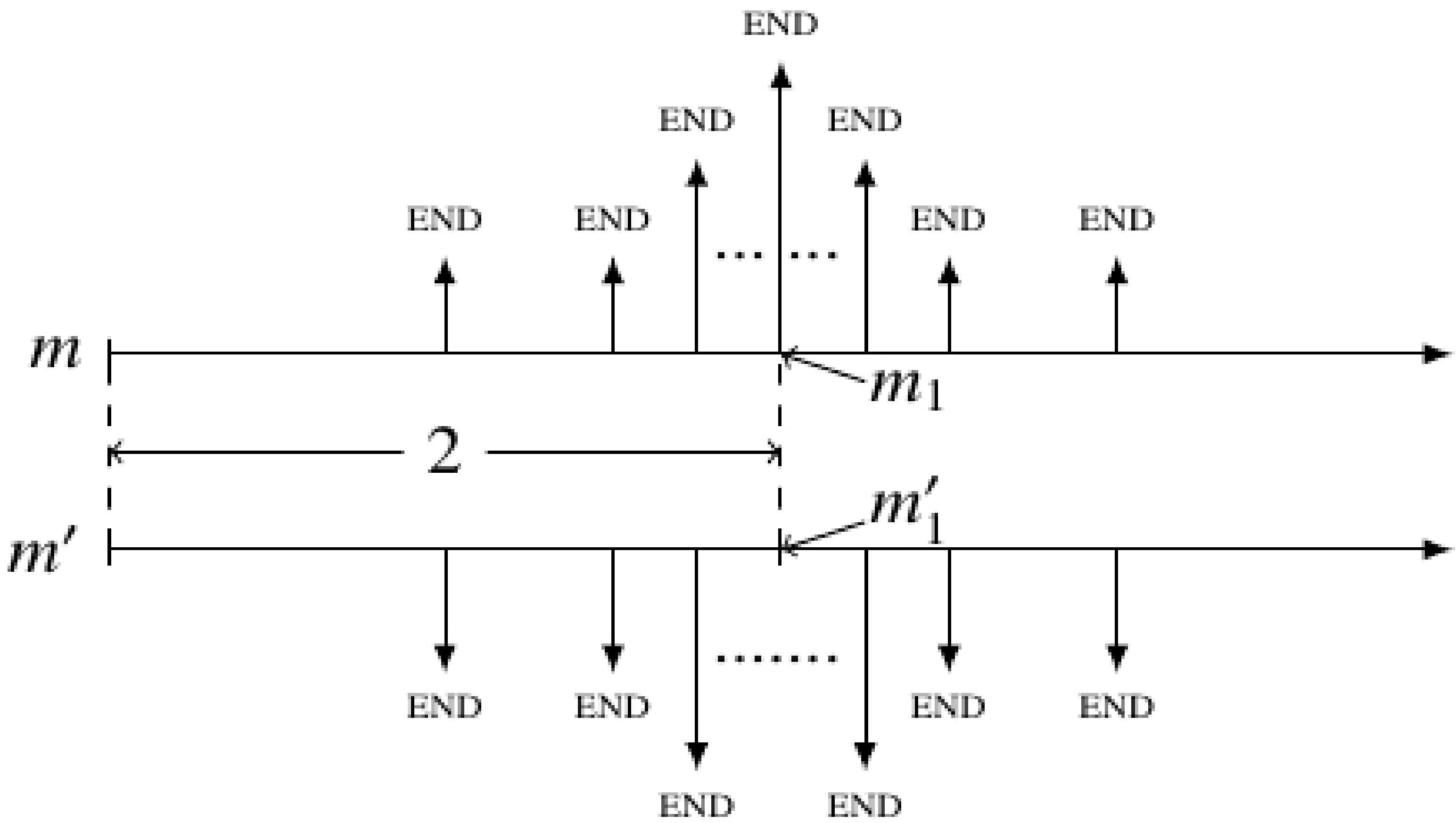
Timed Transition System (TTS)

- $\mathcal{M} = (M, \Sigma, \theta, \oplus)$
  - M: non-empty set of states
  - Σ: non-empty set of actions
  - $\theta: M \times \Sigma \longrightarrow 2^M$
    labeled transition function
  - $\oplus$: $\mathbb{R}_{\geq 0} \times M \rightharpoonup M$
    delay transition function
    - $0 \oplus m = m$;
    - $d \oplus (d' \oplus m) \cong (d + d') \oplus m$

Timed Bisimulation

- $R \subseteq M \times M$ s.t. whenever $(m_1, m_2) \in R$:
  - If $m_1 \xrightarrow{a} m_1'$, then $\exists m_2' \in M$ s.t. $m_2 \xrightarrow{a} m_2'$, and $(m_1', m_2') \in R$;
  - If $d \oplus m_1$ defined, then $\mathrm{d} \oplus m_2$ defined, and $(d \oplus m_1, d \oplus m_2) \in R$.

Timed Bisimulation



Timed Modal Logic (TML)

- Syntax

$\mathcal{L}$: $\phi ::= \bot \mid x \trianglelefteq r \mid \phi \to \phi \mid [a]\phi \mid ₩\phi \mid \forall x.\phi$

where: $r \in \mathbb{Q}_{\geq 0}, \trianglelefteq \in \{\leq, \geq\}, x \in \mathcal{K}$

- Semantics:
  - $M, m, i \vDash \phi$

where $i: \mathcal{K} \to \mathbb{R}_{\geq 0}$

Timed Modal Logic (TML)

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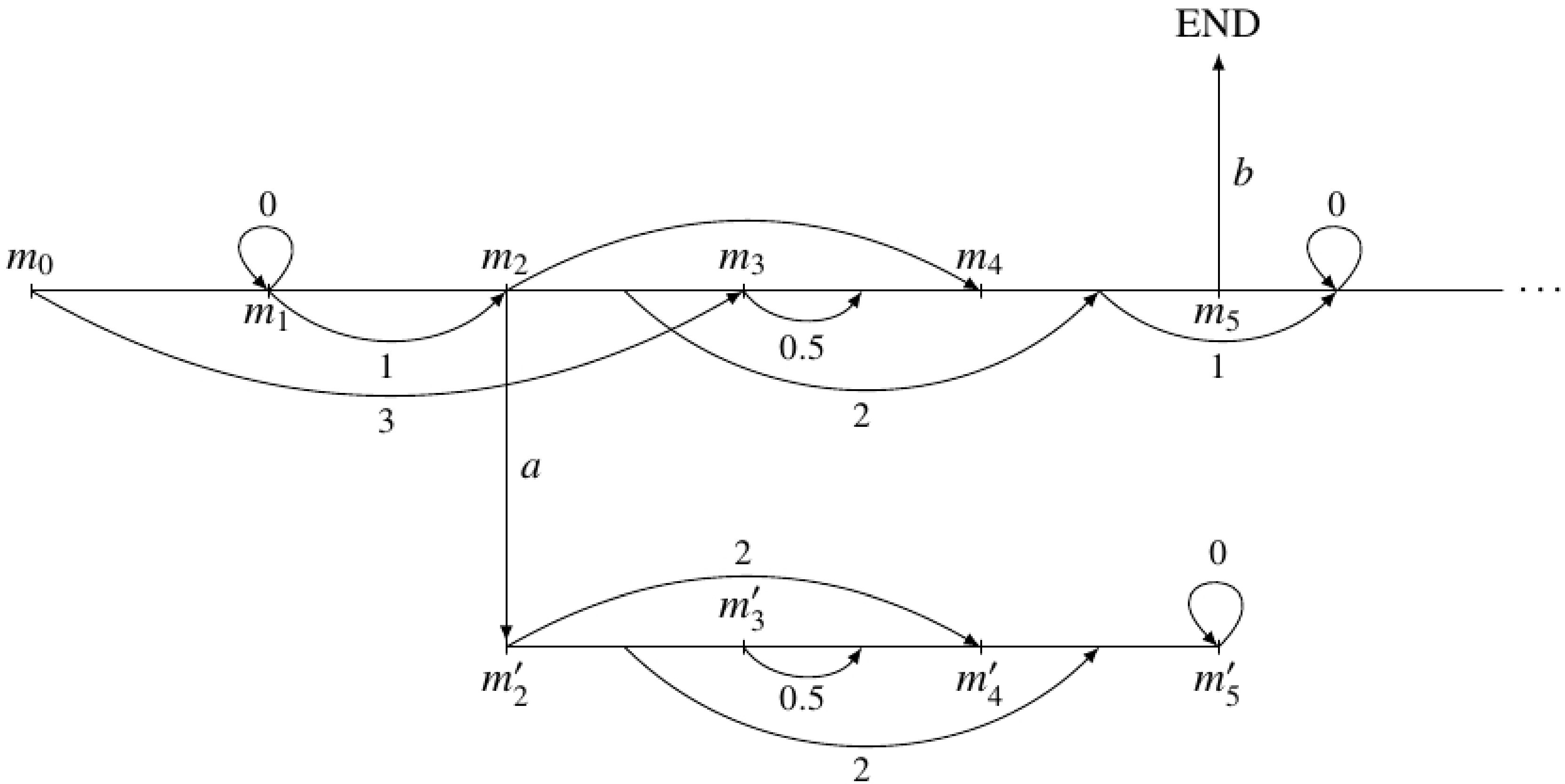
  where: $r \in \mathbb{Q}_{\geq 0}, \trianglelefteq \in \{\leq, \geq\}, x \in \mathcal{K}$

- Semantics:
  - $M, m, i \vDash x \trianglelefteq r$ iff $i(x) \trianglelefteq r$

Timed Modal Logic (TML)

- $i(x) = 2$
- $M, m_0, i \vDash x \geq 2$



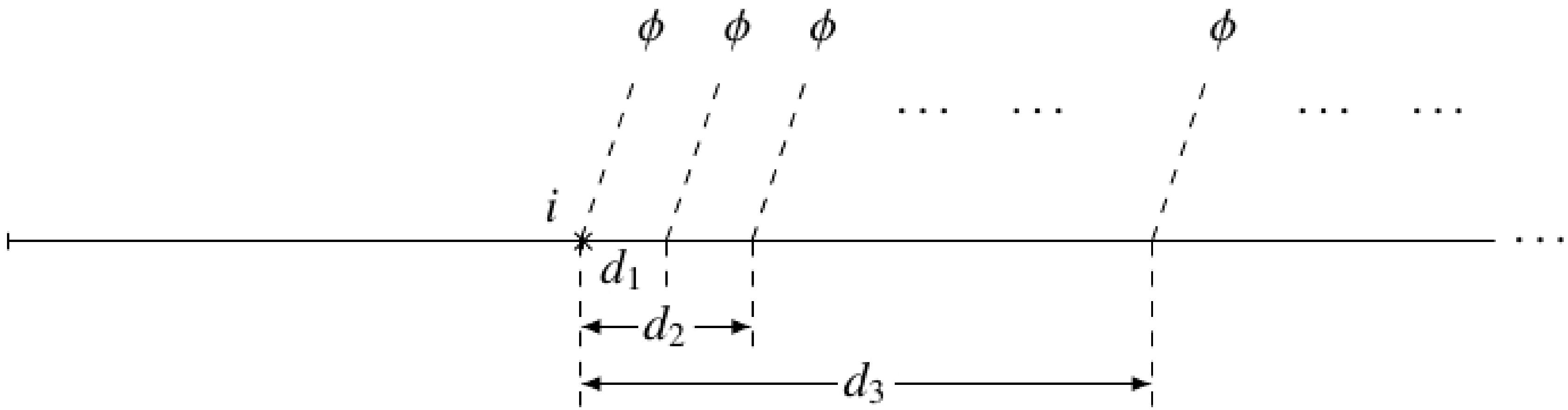
Timed Modal Logic (TML)

- Syntax

  $\mathcal{L}$: $\quad \phi ::= \bot \mid x \trianglelefteq r \mid \phi \to \phi \mid [a]\phi \mid \text{₩}\phi \mid \forall x.\phi$

- Semantics:
  - $M, m, i \vDash \text{₩}\phi$ iff

    for any $d \in \mathbb{R}_{\geq 0}$ $s.t. m' = d \oplus m, M, m', i + d \vDash \phi$



Timed Modal Logic (TML)

- Syntax

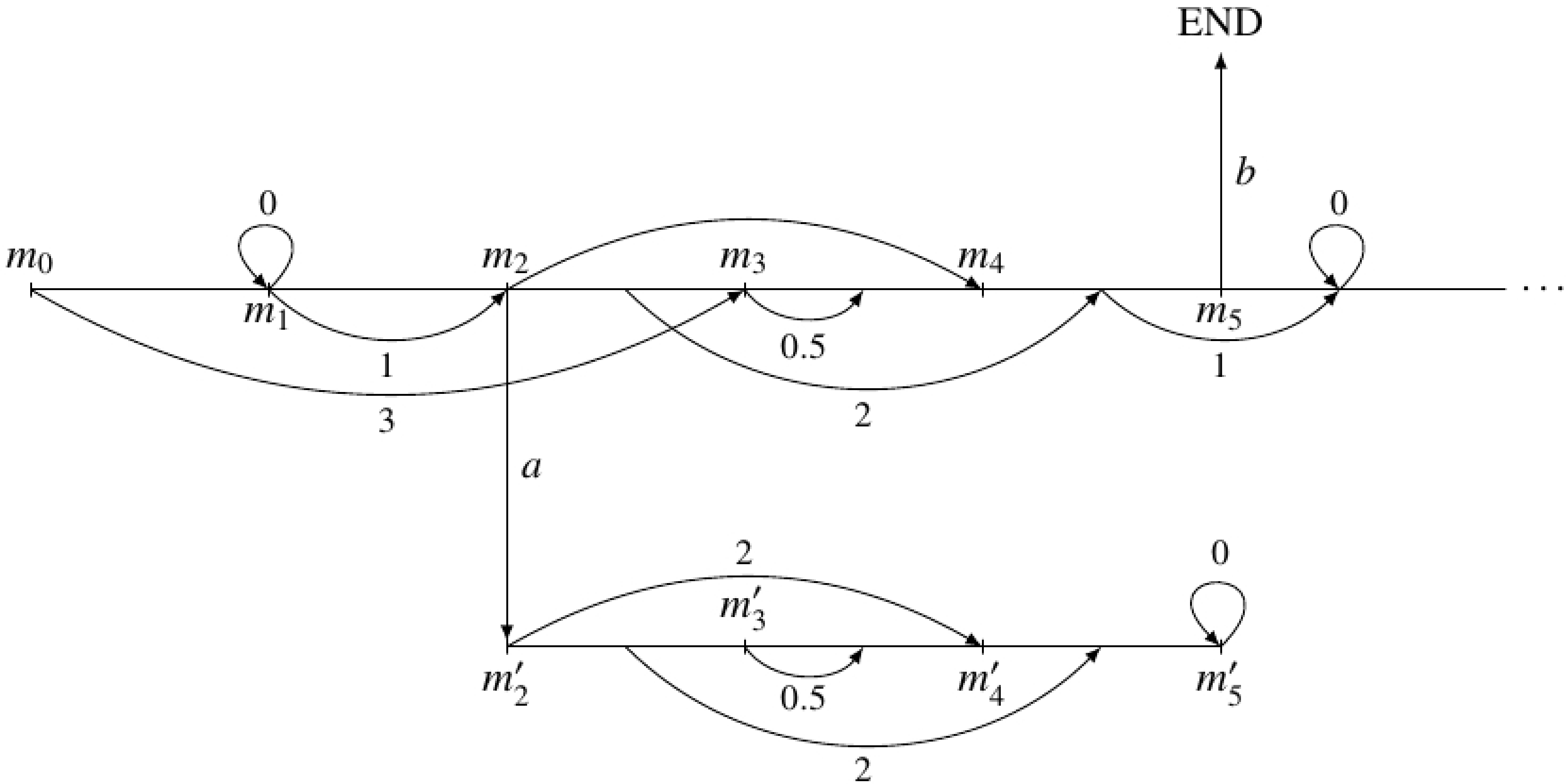
  $\mathcal{L}$: $\phi ::= \bot \mid x \trianglelefteq r \mid \phi \rightarrow \phi \mid [a]\phi \mid ₩\phi \mid \forall x.\phi$

- Semantics:
  - $∃\phi = \neg(₩(\neg\phi))$
  - $M, m, i \vDash ∃\phi$ iff

    there exists $d \in \mathbb{R}_{\geq 0}$ $s.t.$ $m' = d \oplus m$ and $M, m', i + d \vDash \phi$

Timed Modal Logic (TML)

- $i(x) = 2$
- $M, m_0, i \models \forall\!\!\!\forall (x \leq 4 \rightarrow \exists\!\!\!\exists \langle a \rangle \top)$



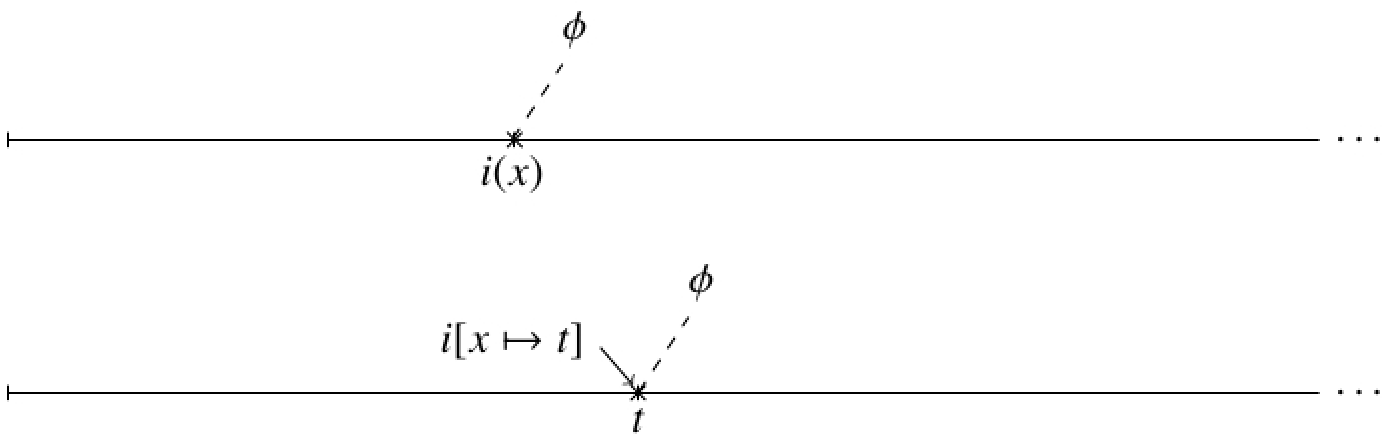
Timed Modal Logic (TML)

- Syntax

  $\mathcal{L}$: $\phi ::= \bot \mid x \trianglelefteq r \mid \phi \to \phi \mid [a]\phi \mid$ ₩ $\phi \mid \forall x.\phi$

- Semantics:
  - $M, m, i \vDash \forall x.\phi$ iff

    for any $t \in \mathbb{R}_{\geq 0}, M, m, i[x \mapsto t] \vDash \phi$



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Timed Modal Logic (TML)

- Syntax

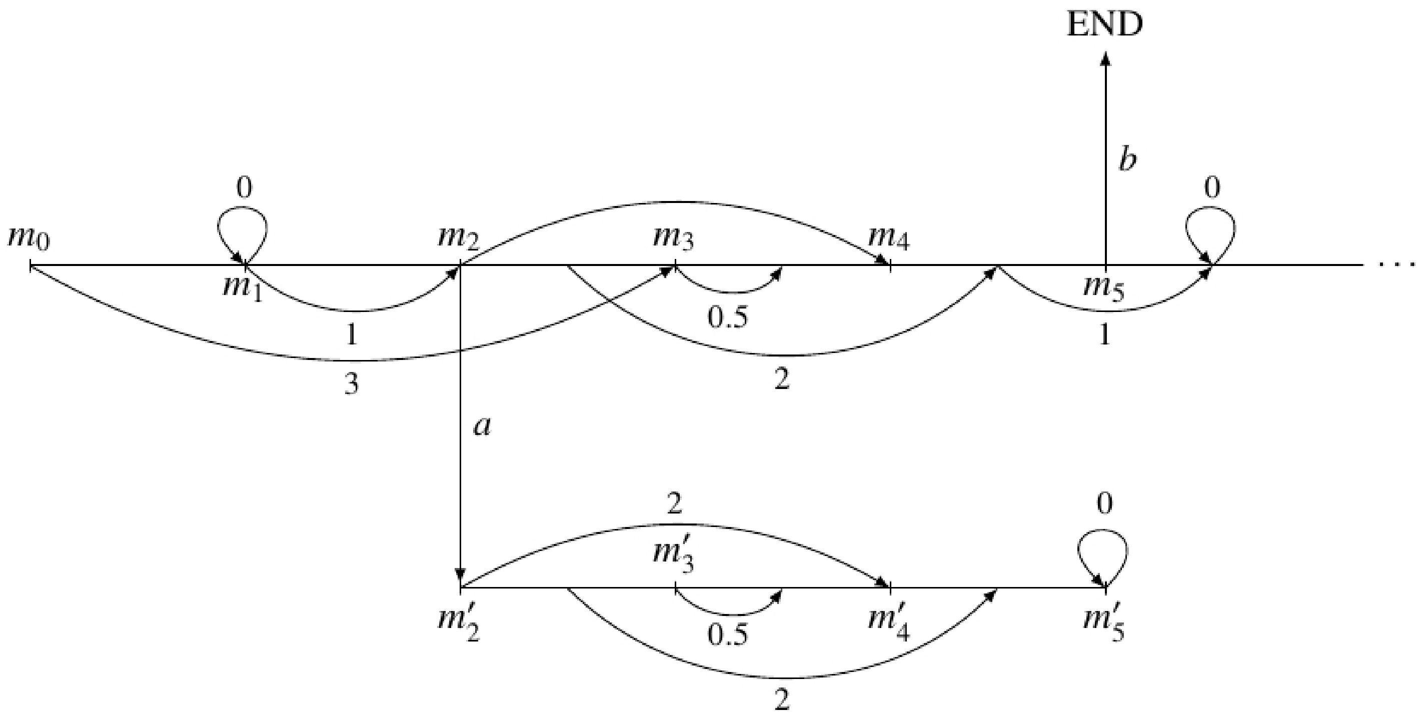
  $\mathcal{L}$: $\phi ::= \bot \mid x \trianglelefteq r \mid \phi \rightarrow \phi \mid [a]\phi \mid \forall\!\!\!\forall\, \phi \mid \forall x.\, \phi$

- Semantics:
  - $\exists x.\phi = \neg(\forall x. \neg\phi)$
  - $M, m, i \vDash \exists x.\phi$ iff

    there exists $t \in \mathbb{R}_{\geq 0}$ s.t. $M, m, i[x \mapsto t] \vDash \phi$

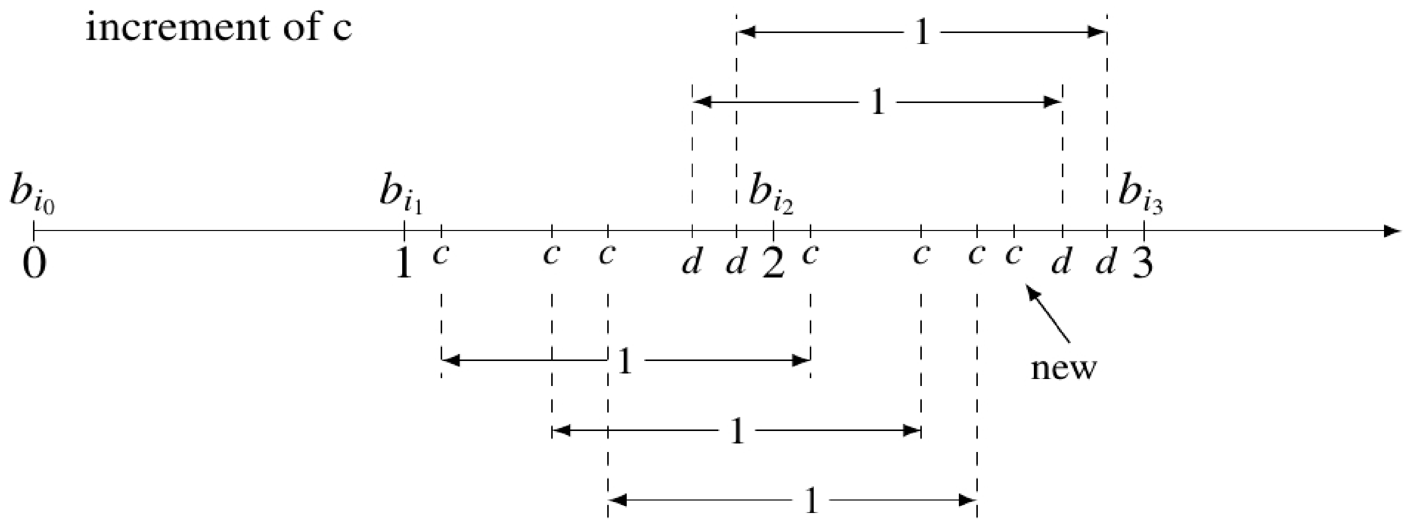
Timed Modal Logic (TML)

- $i(x) = 2$
- $M, m_0, i \vDash \forall x. (x = 0 \rightarrow \forall\!\!\!\forall (x \leq 2 \rightarrow \exists\!\!\!\exists \langle a \rangle \top))$



Undecidability

- Theorem
  - The satisfiability question for TML is Σ_1^1-hard, hence undecidable.
  - Proof: encode a non-deterministic 2-counter machine



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Undecidability

- Theorem
  - The satisfiability question for TML is Σ_1^1-hard, hence undecidable.
  - The set of TML-validities is not RE

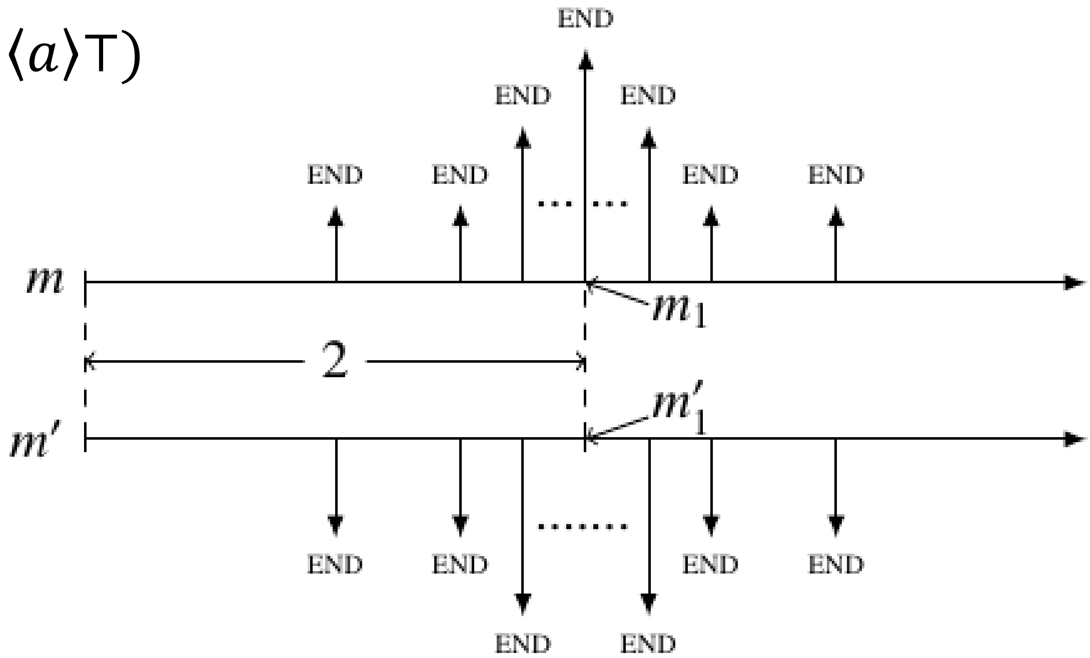
Adequacy of TML

- Adequacy of modal logics:
  - $m \sim m'$ iff for any ϕ,

$$M, m \vDash \phi \Leftrightarrow M, m' \vDash \phi$$

Adequacy of TML

- $i(x) = \pi - 3$, for any ϕ:
  $M, m, i \vDash \phi$ iff $M', m', i \vDash \phi$
- $i'(x) = 0$,
  $\psi = \exists\!\!\exists(x = 2 \wedge \langle a \rangle \top)$
  $M, m, i' \vDash \psi$
  $M', m', i' \nvDash \psi$



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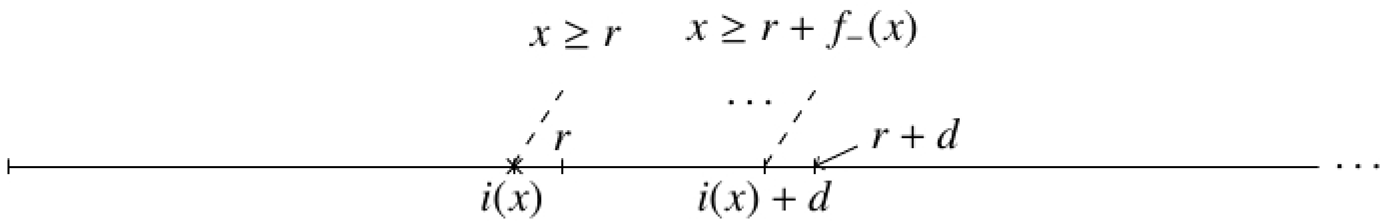
Adequacy of TML

- Changing the interpretations



where $f_{-}(x) \leq d \leq f_{+}(x)$

- $\phi + {}^{f_{-}}/_{f_{+}}$
- $\phi +_{\sigma} {}^{f_{-}}/_{f_{+}}$
- $\phi + f$

Adequacy of TML

- Lemma:
  - $M, m, i \vDash \phi \Longrightarrow M, m, (i + \delta) \circ \sigma^{-1} \vDash \phi +_{\sigma} {}^{f_-}/_{f_+}$
- Corollary:
  - $M, m, i \vDash \phi \Leftrightarrow M, m, i + f \vDash \phi + f$
  - $M, m, i \vDash \phi \Leftrightarrow M, m, i + \delta \vDash \phi$

Adequacy of TML

- Theorem:
  - $m \sim m'$ iff for any i and ϕ,

$$M, m, i \vDash \phi \Leftrightarrow M, m', i \vDash \phi$$

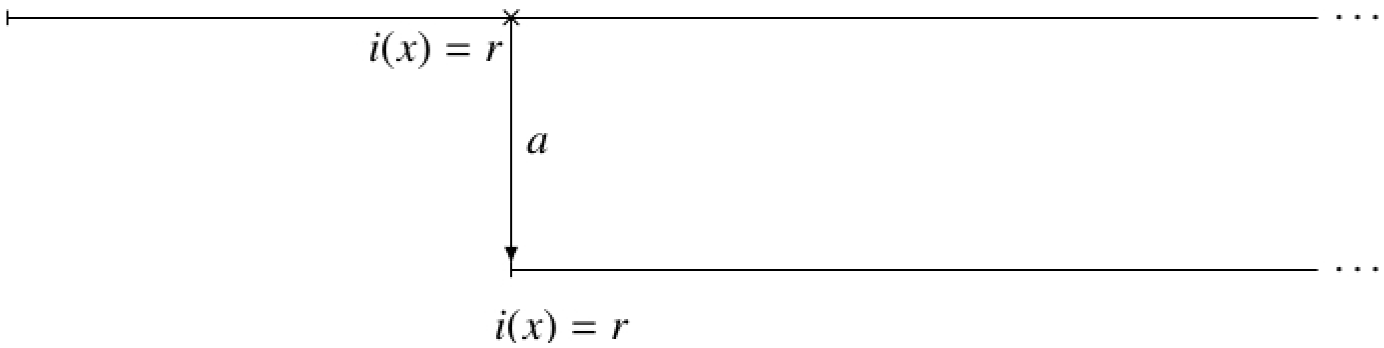
Axiomatization for TML

- Axioms and Rules
  - $\vdash \Box(\phi \rightarrow \psi) \rightarrow (\Box\phi \rightarrow \Box\psi)$
  - If $\vdash \phi$, then $\vdash \Box\phi$

    where $\Box \in \{[a],\ \forall\!\!\forall,\ \forall x.\}$

Axiomatization for TML

- Axioms and Rules
  - $\vdash x \trianglelefteq r \to [a](x \trianglelefteq r)$



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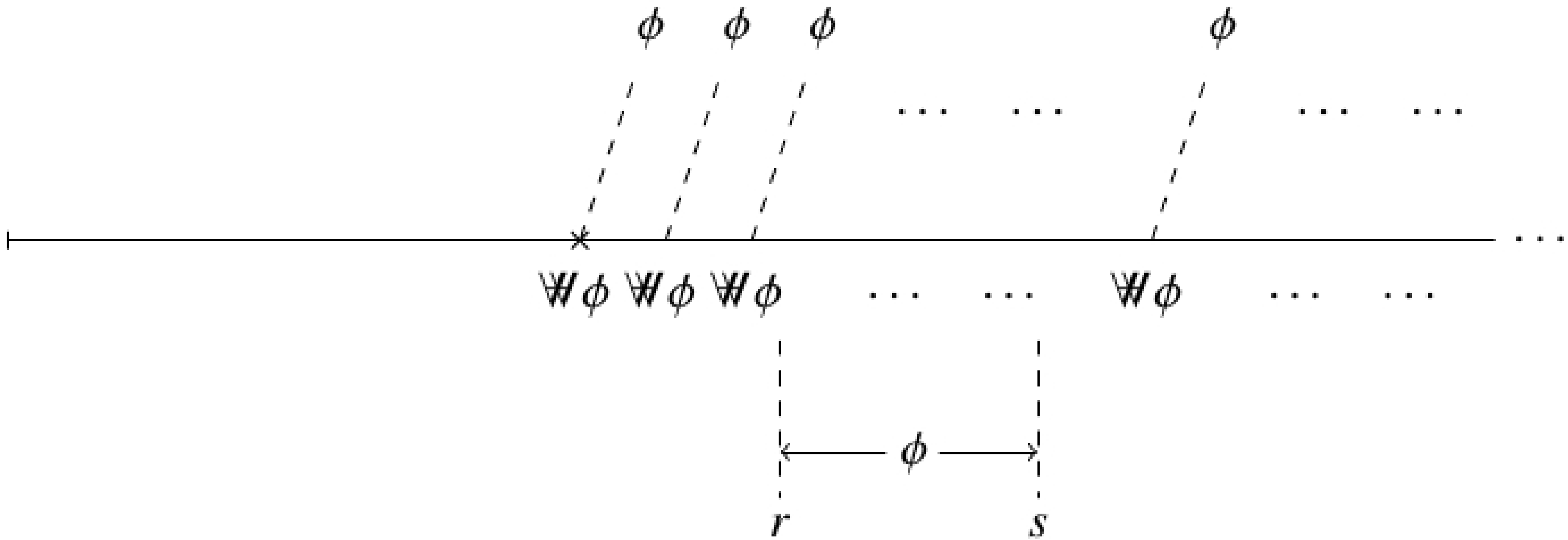
Axiomatization for TML

- Axioms and Rules
  - $\vdash \text{₩}\phi \rightarrow \phi$
  - $\vdash \text{₩}\phi \rightarrow \text{₩₩}\phi$
  - $\vdash \text{₩}\phi \rightarrow \text{₩}(r \leq x \leq s \rightarrow \phi), r \leq s$



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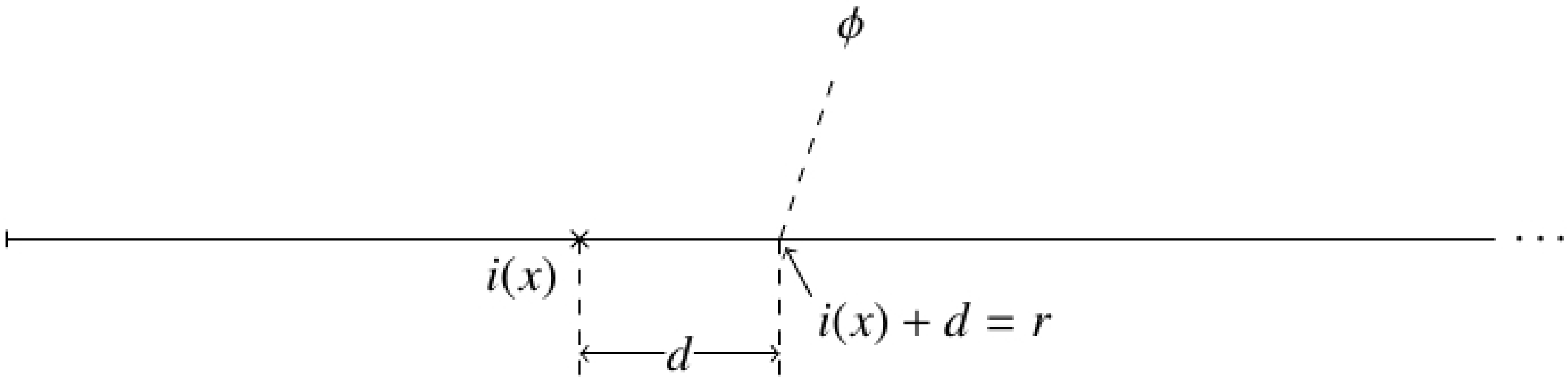
Axiomatization for TML

- Axioms and Rules
  - $\vdash \exists\!\!\exists (x = r \wedge \phi) \rightarrow \forall\!\!\!\forall (x = r \rightarrow \phi)$



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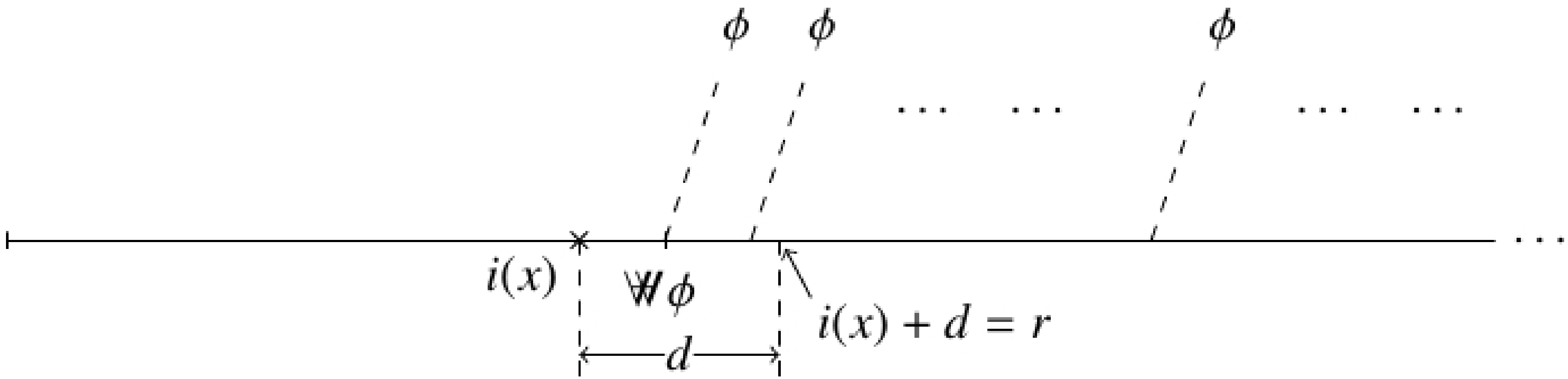
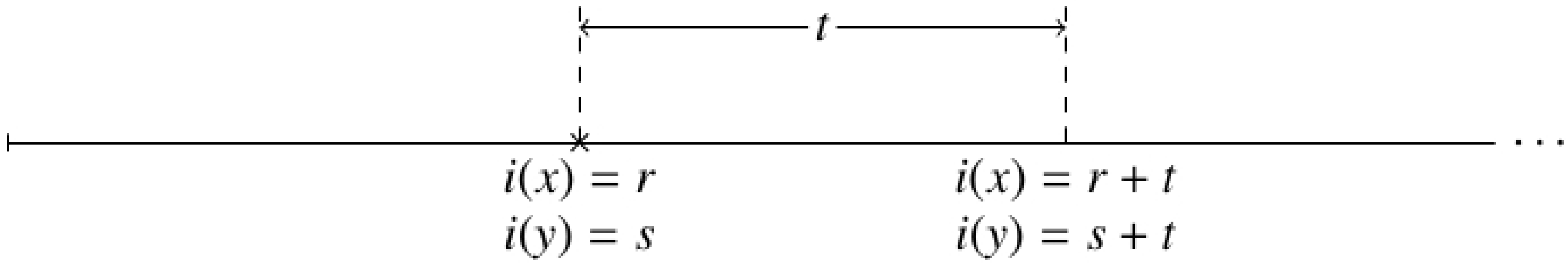
Axiomatization for TML

- Axioms and Rules
  - $\vdash \exists\!\!\exists\,(x \leq r \wedge \forall\!\!\forall \phi) \rightarrow \forall\!\!\forall\,(x \geq r \rightarrow \phi)$



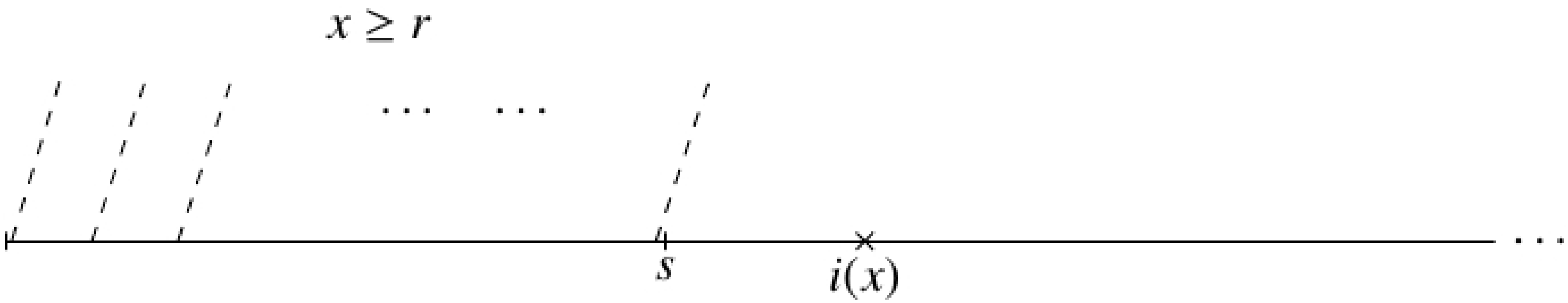
Axiomatization for TML

- Axioms
  - $\vdash x \trianglelefteq r \wedge y \trianglelefteq s \rightarrow \forall(x \trianglelefteq r + t \rightarrow y \trianglelefteq s + t)$



Axiomatization for TML

- Infinitary Rules
  - $\{x \trianglelefteq r \mid r \rhd s\} \vdash x \trianglelefteq s$
  - $\{x \geq r \mid r \in \mathbb{Q}_{\geq 0}\} \vdash \bot$



Axiomatization for TML

- Infinitary Rules
  - $\{x \trianglelefteq r \mid r \triangleright s\} \vdash x \trianglelefteq s$
  - $\{x \geq r \mid r \in \mathbb{Q}_{\geq 0}\} \vdash \bot$
  - $\left\{\phi + {}^{[x \mapsto r]}/_{[x \mapsto s]} \,\middle|\, r \leq s\right\} \vdash \forall x.\phi$
  - $\{ ∀\!\!\!\forall (x \leq s \rightarrow \phi) \mid s \in \mathbb{Q}_{\geq 0}\} \vdash ∀\!\!\!\forall \phi$

Axiomatization for TML

- Infinitary Rules
  - $\{C[x \trianglelefteq r] \mid r \rhd s\} \vdash C[x \trianglelefteq s]$
  - $\{C[x \geq r] \mid r \in \mathbb{Q}_{\geq 0}\} \vdash C[\bot]$
  - $\{C\left[\phi + {}^{[x \mapsto r]}/_{[x \mapsto s]}\right] \mid r \leq s\} \vdash C[\forall x. \phi]$
  - $\{C[\text{₩}(x \leq s \to \phi)] \mid s \in \mathbb{Q}_{\geq 0}\} \vdash C[\text{₩}\phi]$

  where C is context: e.g.,

  $$\varepsilon[X], [a]X, \forall x. \text{₩}X, \forall x. [a]\text{₩}[b][c]X$$

❖ *Kozen, Larsen, Mardare, Panangaden (LICS 2013)*

Axiomatization for TML

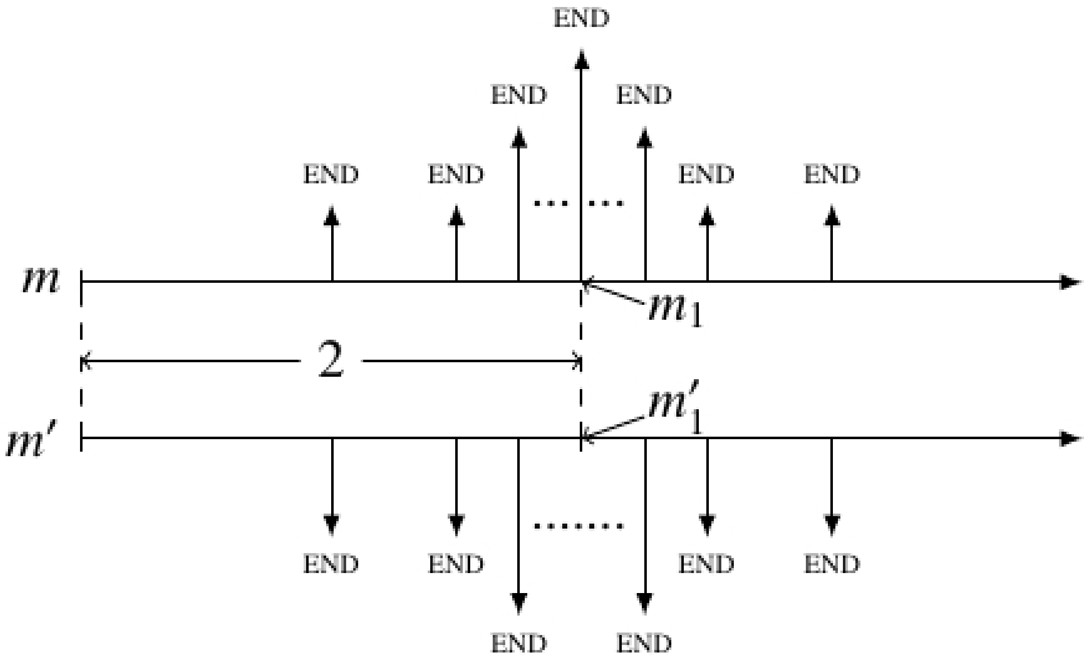
- Infinitary Rules
  - $\{C[x \trianglelefteq r] \mid r \rhd s\} \vdash C[x \trianglelefteq s]$
  - $\{C[x \geq r] \mid r \in \mathbb{Q}_{\geq 0}\} \vdash C[\bot]$
  - $\{C\left[\phi + {}^{[x \mapsto r]}/_{[x \mapsto s]}\right] \mid r \leq s\} \vdash C[\forall x. \phi]$
  - $\{C[\forall\!\!\!\forall(x \leq s \rightarrow \phi)] \mid s \in \mathbb{Q}_{\geq 0}\} \vdash C[\forall\!\!\!\forall \phi]$
- Non-compactness
  - $\Phi = \{x \geq r \mid r < s\} \cup \{x < s\}$

Axiomatization for TML

- Soundness:
  - $\Phi \vdash \phi \Longrightarrow \Phi \models \phi$
- Completeness:
  - Weak-completeness
    - $\models \phi \Longrightarrow \vdash \phi$
  - Strong-completeness
    - $\Phi \models \phi \Longrightarrow \Phi \vdash \phi$

Axiomatization of TML

- $\psi = \exists\!\!\exists\,(x = 2 \wedge \langle a \rangle \top)$
  - $i(x) = \pi - 3$,
    $M, m, i \nvDash \psi$
  - $i'(x) = 0$,
    $M, m, i' \vDash \psi$



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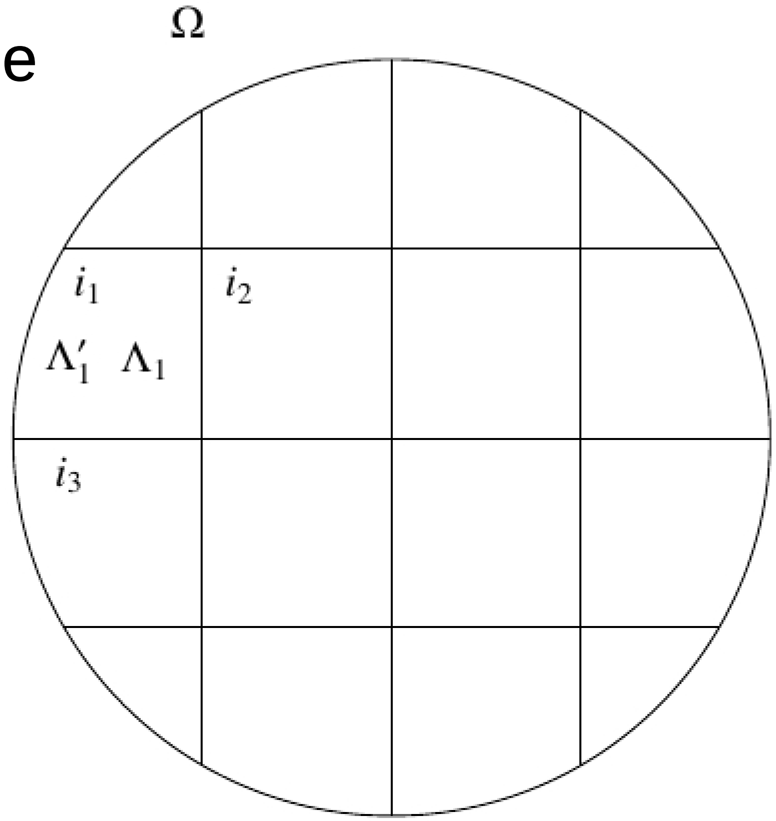
Axiomatization for TML

- Lemma
  - For any maximal consistent set $\Lambda \in \Omega$, there is only one i s.t. $\mathcal{I}(\Lambda) = i$.



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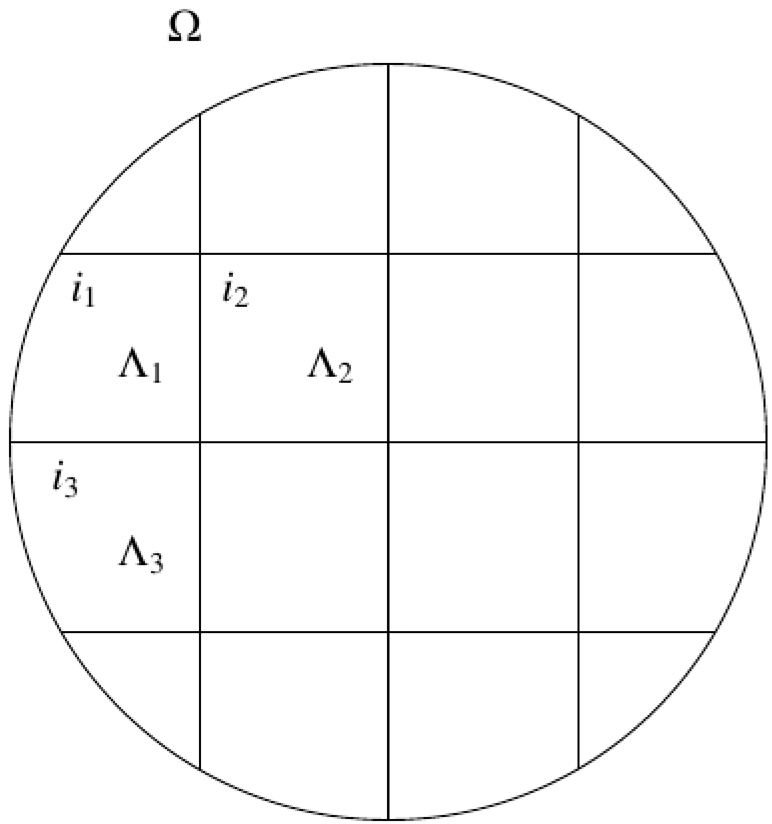
Axiomatization for TML

- Lemma
  - For any $\phi \in \Lambda_1$,

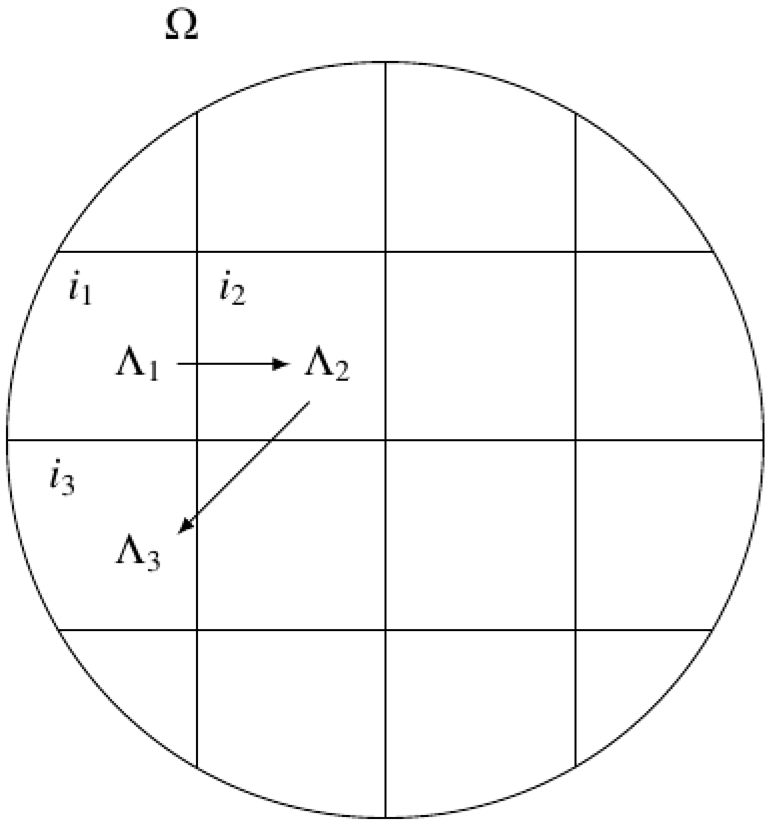
    $\phi +_{\sigma} {}^{f_-}/_{f_+} \in \Lambda_2$

    where $f_- \leq i_2 - i_1 \leq f_+$



Axiomatization for TML

- Lemma
  - $\gamma\colon \mathcal{I} \to \Omega$
  - For any $\Lambda \in \Omega$, there exists γ s.t. $\gamma\big(\mathcal{I}(\Lambda)\big) = \Lambda$.



Axiomatization for TML

- Canonical model
  - $\Gamma = (\Gamma, \theta, \oplus)$
    - $\Gamma = \{\gamma \mid \gamma \text{ is a coherent function}\}$
    - $\theta: \gamma \xrightarrow{a} \gamma'$ if for any i, $[a]\phi \in \gamma(i) \Rightarrow \phi \in \gamma'(i)$
    - $\gamma' = d \oplus \gamma$ if for any i, $\text{₩}\ \phi \in \gamma(i) \Rightarrow \phi \in \gamma'(i+d)$

Axiomatization for TML

- Truth Lemma
  - $\Gamma, \gamma, i \vDash \phi$ iff $\phi \in \gamma(i)$
- Strong completeness
  - $\Phi \vDash \phi \Longrightarrow \Phi \vdash \phi$

Conclusion and Future work

- Conclusion
  - Adequacy of TML
  - Undecidability of satisfiability problem for TML
  - Strong-complete axiomatization

- Future work
  - Stone duality extend to this setting?
  - Extension of Boolean Algebra which represents TML

Timed Modal Logic

- Thanks!