Conditional Independence, Computability, and Measurability

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Emmanuel College
University of Cambridge

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Algorithmic processes that describe and transform uncertainty.
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The stochastic inference problem (informal version)

Input: guesser and checker probabilistic programs.
Output: a sample from the same distribution as the program

\begin{verbatim}
accept = False
while (not accept):
    guess = guesser()
    accept = checker(guess)
return guess
\end{verbatim}

This computation captures Bayesian statistical inference.

- "prior" distribution $\leftrightarrow$ distribution of \( \text{guesser()} \)
- "likelihood(\( g \))" $\leftrightarrow$ \( \Pr(\text{checker}(g) \text{ is True}) \)
- "posterior" distribution $\leftrightarrow$ distribution of return value
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Example: predicting next coin toss in a sequence

Let $n \geq 0$ and $x_1, \ldots, x_n \in \{0, 1\}$. E.g., 0, 0, 1, 0, 0, 0, ?

guesser():
• sample $\theta$ and $U$ independently and uniformly in $[0, 1]$, and
• return $(\theta, X)$ where $X = 1(U \leq \theta)$.

checker($\theta$, $x$):
• sample $U_1, \ldots, U_n$ independently and uniformly in $[0, 1]$, and
• let $X_i = 1(U_i \leq \theta)$, and
• accept if and only if $X_i = x_i$ for all $i$. 

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- let $X_i = 1(U_i \leq \theta)$, and 
- accept if and only if $X_i = x_i$ for all $i$. 

Let $s = x_1 + \cdots + x_n$ and let $U$ be uniformly distributed. For all $t \in [0, 1]$, we have $\Pr(U \leq t) = t$ and

$$\Pr(\text{checker}(t, x) \text{ is True}) = \Pr(\forall i \ (U_i \leq t \iff x_i = 1)) = t^s(1-t)^{n-s}.$$  

$n = 6, \ s \in \{1, 3, 5\}.$

$$\Pr(\text{checker}(U, x) \text{ is True}) = \int_0^1 t^s(1-t)^{n-s} \, dt = \frac{(s)!(n-s)!}{(n+1)!} =: Z(s)$$

Let $p(t)\,dt$ be the probability that the accepted $\theta \in [t, t + dt)$.

$$p(t)\,dt \approx t^s(1-t)^{n-s} \, dt + (1 - Z(s))p(t)\,dt \approx \frac{t^s(1-t)^{n-s}}{Z(s)} \, dt$$

Probability that the accepted $X = 1$ is then $\int t \, p(t)\,dt = \frac{s+1}{n+2}$. 
Example: fitting a line to data (aka linear regression)

```python
accept = False
while (not accept):
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```

Let \((x_i, y_i) \in \mathbb{R}^2\) and \(\sigma, \nu, \varepsilon > 0\).

guesser():
- sample coefficients \(\alpha, \beta\) independently from \(\text{Normal}(0, \sigma^2)\).

checker(\(\alpha, \beta\)):
- sample independent noise variables \(\xi_i\) from \(\text{Normal}(0, \nu^2)\),
- let \(F(x) = \alpha x + \beta\) and \(Y_i = F(x_i) + \xi_i\), and
- accept if and only if \(|Y_i - y_i| < \varepsilon\) for all \(i\).
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Note that \(\varepsilon = 0\) doesn’t work, but the limit \(\varepsilon \to 0\) makes sense.
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Fantasy example: extracting 3D structure from images

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Example: not so fantastical [Mansinghka et al.]
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Let $U$ be a Uniform$(0, 1)$ random variable.
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Let $U$ be a Uniform$(0, 1)$ random variable.
Let $S$ and $T$ be a computable metric space.
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Let $S$ and $T$ be a computable metric space.

**INPUT:**

- $X : [0, 1] \rightarrow S$,
- $Y : [0, 1] \rightarrow T$, and
- $x \in S$. 
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**INPUT:**
- $X : [0, 1] \rightarrow S$,
- $Y : [0, 1] \rightarrow T$, and
- $x \in S$.

**OUTPUT:**
- a sample from $\Pr(Y(U)|X(U) = x)$,
i.e., the conditional distribution of $Y(U)$ given $X(U) = x$. 
Bayesian statistics

1. Express statistical assumptions via probability distributions.

\[
\Pr(\text{parameters, data}) = \Pr(\text{parameters}) \Pr(\text{data | parameters})
\]

2. Statistical inference from data → parameters via conditioning.

\[
\Pr(\text{parameters, data}), \ x \xrightarrow{\text{conditioning}} \Pr(\text{parameters | data = x})
\]

Probabilistic programming

1. Represent probability distributions by formulas probabilistic programs that generate samples.
2. Build generic algorithms for probabilistic conditioning using probabilistic programs as representations.
Talk Outline

1. The stochastic inference problem
2. Where are we now in probabilistic programming?
3. Approximability and Exchangeability:
   When can we represent conditional independence?
4. Conclusion
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Questions raised

- Which operations in probability theory can we perform when distributions are represented by programs?
- When can we perform these computations efficiently?
- How are statistical properties (e.g., symmetries) of a distribution reflected in the structure of the computation representing it?
Q: Can we automate conditioning?

\[ \Pr(X, Y), \ x \xrightarrow{} \Pr(Y|X = x) \]

A: No, but almost.

---

[X discrete] [\( \square \)] [\( \times \)] [\( \square \)] [\( \square \)]

\[ \Pr(\xi) \text{ smooth} \]

\[ p(X|S) \text{ given} \]

[Freer and R., 2010] [Ackerman, Freer, and R., 2011] ...
Q: What about **EFFICIENT** inference?

\[
\Pr(X, Y), \ x \rightarrow \Pr(Y|X = x)
\]

A: It’s complicated.

---

```python
def hash_of_random_string(n):
    str = random_binary_string(n)
    return cryptographic_hash(str)
```

---

[Diagram: A Bayes net with nodes X and Y, and an edge from X to Y, labeled with X discrete.]
Q: What about EFFICIENT inference?

\[ \Pr(X, Y), \ x \mapsto \Pr(Y|X = x) \]

A: It’s complicated.

\[
\begin{align*}
X & \quad \text{discrete} \\
Y \quad & \quad \\
\downarrow \quad & \\
X & \quad \text{X} \\
\end{align*}
\]

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Q: What explains the success of probabilistic methods?
A: Structure like conditional independence.

- Bayes nets are representations of distributions that expose conditional independence structure via a directed graph.
- The complexity of exact inference in Bayes nets is controlled by the tree width of the graph.
Q: Are probabilistic programs sufficiently general as representations for stochastic processes?

We are missing a notion of approximation!

Theorem (Avigad, Freer, R., and Rute).

“Approximate samplers can represent conditional independencies that exact samplers cannot.”
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Concrete example of an exchangeable sequence

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Sum = 1.0; Total = 2.0
def next_draw():
    global Sum, Total
    y = bernoulli(Sum/Total)
    Sum += y; Total += 1
    return y
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>>> repeat(next_draw, 10)
[0, 1, 1, 0, 1, 1, 1, 0, 1, 0]
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![Graph showing the sum and total values over iterations.](image)

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![Graphs showing the increase in Sum and Total over iterations](image)
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\text{Sum} &= 1.0; \quad \text{Total} = 2.0 \\
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\text{theta} &= \text{uniform(0,1)} \\
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2 &\quad 3 \\
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5 &\quad 6 \\
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8 &\quad 9 \\
9 &\quad 10
\end{align*}
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**Definition.** A sequence $Y = (Y_1, Y_2, \ldots)$ of random variables is exchangeable when

$$ (Y_1, \ldots, Y_n) \overset{d}{=} (Y_{\pi(1)}, \ldots, Y_{\pi(n)}), $$

(1)

for all $n \in \mathbb{N}$ and permutation $\pi$ of $\{1, \ldots, n\}$.
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**Theorem (de Finetti).** The following are equivalent:

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Informally: using $f$, we can sample $Y_i$’s in parallel.
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3. Exists $f$ such that

$$ (Y_1, Y_2, Y_3, \ldots) \overset{d}{=} (f(\theta, U_1), f(\theta, U_2), f(\theta, U_3), \ldots) $$

for i.i.d. uniform $\theta, U_1, U_2, \ldots$. 

Informally: using $f$, we can sample $Y_i$'s in parallel.
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Informally: using $f$, we can sample $Y_i$’s in parallel.
We can extract the hidden parallelism. [Freer and R., 2012]

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Sum = 1.0; Total = 2.0
def next_draw():
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theta = uniform(0,1)
def next_draw():
    return bernoulli(theta)

Theorem (Freer and R., 2012). The distribution of an exchangeable sequence \( Y \) is computable if and only if there is an almost computable \( f \) such that \((Y_1, Y_2, \ldots) \sim (f(\theta, U_1), f(\theta, U_2), \ldots)\).

We can always recover hidden parallel structure, exposing conditional independence to the inference engine.
We can extract the hidden parallelism. [Freer and R., 2012]

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We can always recover hidden parallel structure, exposing conditional independence to the inference engine.
Where else can we find hidden conditional independence?

Can we extract it for inference?
Exchangeable arrays in models of graph/relational data

**Definition.**

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<tr>
<th>structure</th>
<th>symmetry</th>
<th>definition</th>
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<tbody>
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Example.

Adjacency matrix \((X_{i,j})\) of an undirected graph on \(N\).
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**Example.** Adjacency matrix \((X_{i,j})_{i,j} \in \mathbb{N}\) of an undirected graph on \(\mathbb{N}\).

Theorem (Aldous-Hoover). \( \theta, U_i, V_j, W_{i,j} \) all i.i.d. uniform.

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<tbody>
<tr>
<td>array ( (X_{i,j}) )</td>
<td>((X_{i,j}) \overset{d}{=} (X_{\pi(i),\tau(j)}))</td>
<td>((X_{i,j}) \overset{d}{=} (f(\theta, V_i, U_j, W_{i,j})))</td>
</tr>
</tbody>
</table>
Exchangeable arrays in models of graph/relational data

**Theorem (Aldous-Hoover).** \( \theta, U_i, V_j, W_{i,j} \) all i.i.d. uniform.

<table>
<thead>
<tr>
<th>structure</th>
<th>symmetry</th>
<th>representation</th>
</tr>
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<td>sequence ((Y_n))</td>
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Visualization of Aldous-Hoover theorem for exchangeable arrays
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Visualization of Aldous-Hoover theorem for exchangeable arrays
Q: Is the Aldous-Hoover theorem computable?
A: No.

Theorem (Avigad, Freer, R., and Rute). There is an exchangeable array \( X \) with a computable distribution but no a.e. computable \( f \) satisfying Aldous-Hoover.

Even “computationally universal” probabilistic programming languages cannot represent certain conditional independence structure.
Computably-distributed array \( X \), noncomputable \( f \) [AFRR]

The construction (an aliens dating site).

- Let rows/columns represent aliens. \( X_{i,j} = 1 \) means aliens \( i \) and \( j \) are matched.
- Each alien answers an infinitely-long questionnaire.
- Question \( k \in \{1, 2, \ldots \} \) has \( 2^k \) possible answers.
- Aliens hate answering questionnaires, so they answer randomly.
- Two aliens are matched if they agree on ANY question.

Proof sketch.

- Note: \( f \) is “return 1 iff two aliens agree somewhere.”
- \( f \) not a.e. computable \([\text{Topological obstruction.}]\)
- Given two questionnaires, can’t accurately check in finite time.
- The probability of agreeing on any question \( n, n+1, \ldots \) decays.
- Using only first \( n \) questions yields an approximation.

Approximating \( f \) sufficed to sample. Q: The converse?
Computably-distributed array $X$, noncomputable $f$ [AFRR]

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- *(array computably-distributed)*
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Approximating $f$ sufficed to sample. Q: The converse?
Silver lining? $f$ is always “nearly computable”

Let $\mu$ be a computable probability measure.

**Definition.** Say $f$ is **a.e. computable** when we can compute $f$ on a set of $\mu$-measure one.

**Definition (Kriesel-Lacombe (1957), Šanin (1968), Ko (1986)).** Say $f$ is **computably measurable** when,
- uniformly for any $\varepsilon > 0$,
- we can compute $f$ on a set of $\mu$-measure at least $1 - \varepsilon$.

**Theorem (Avigad, Freer, R., and Rute).** The distribution of an exchangeable array $X$ is computable if and only if there is a computably measurable function $f$ satisfying Aldous-Hoover.
Exchangeability and probabilistic programming

Exchangeable random structures possess a lot of structure.

\[ (Y_i) \overset{d}{=} (f(\theta, U_i)) \]
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Can your favorite PPL represent \( f \)?
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But do such arrays appear in practice?
Do such arrays $X$ appear in practice?

YES!

- √ a.e. computable $f$
- × merely computably measurable $f$

√ Infinite Relational Model
  (Kemp, Tenenbaum, Griffiths, Yamada, and Ueda 2008)

√ Linear Relational Model
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Dirichlet process
Mondrian process
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Talk Outline

1. The stochastic inference problem
2. Where are we now in probabilistic programming?
3. Approximability and Exchangeability: When can we represent conditional independence?
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2. How do we use such representations? Exact-approximate inference and computable measurability?
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