Towards a quantum domain theory

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MFPS XXX

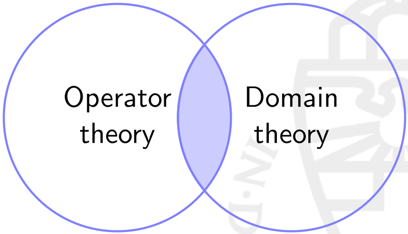
Operator theory vs. Domain theory

Order-theoretic properties of operator algebras

- 1946 J.-P. Vigier
- 1951 J. Dixmier
- 1951 R. V. Kadison
- 1951 S. Sherman
- 1956 R. V. Kadison
- 2007 K. Saito, J.D. Maitland Wright

Semantics of quantum programming language

- 2004 P. Selinger
- 2006 E. D'Hondt, P. Panangaden



An interesting connection is emerging!

Current semantical interpretation of quantum types

$A, B := I \mid A \otimes B \mid 0 \mid A + B \mid bit \mid qbit$

- $[\![bit]\!] = \mathbb{C}^2$, $[\![trit]\!] = \mathbb{C}^3$, ...
- $[\![qbit]\!] = M_2(\mathbb{C}), [\![qtrit]\!] = M_3(\mathbb{C}), ...$
- $[\![0]\!] = 0 = \{0\}$
- $[\![I]\!] = \mathbb{C}$
- $[\![A + B]\!] = [\![A]\!] \oplus [\![B]\!]$
- $[\![A \otimes B]\!] = [\![A]\!] \otimes [\![B]\!]$

Typed programs = completely positive maps

What about **recursive** types and **infinite** types?

Results in this paper

1. Hom-sets of normal (positive) sub-unital maps between W*-algebras are **directed-complete**.
2. $\mathbf{W}^*$, category of W*-algebras together with normal sub-unital maps, is **order-enriched**.
3. $\mathbf{W}^*$ is **algebraically compact** for an important class of functors.
   - This means: these functors have both initial algebras and final coalgebras, and they coincide.
   - Peter Freyd, "Remarks on algebraically compact categories", 1992.
   - Michael Barr, "Algebraically compact functors", 1995.

Plan

Hilbert spaces and W*-algebras

- Hilbert spaces are used in quantum foundations
- Examples: $\mathbb{C}^n$, $\ell^2(\mathbb{N})$, ...
- $\mathcal{B}(H)$ = collection of bounded linear operators on a Hilbert space H. $\mathcal{B}(H)$ has a multiplication and an involution
- W*-algebra: special subalgebra of $\mathcal{B}(H)$ closed in uniform convergence, pointwise convergence, the structure of $\mathcal{B}(H)$ and its order; algebra of observables in quantum theory.

Example

1. $\mathcal{B}(H)$: collection of bounded operators on a Hilbert space H
2. $L^\infty(X)$: function space for some standard measure space X (commutative)
3. $\ell^\infty(\mathbb{N})$: space of bounded sequences (commutative and separable)

Maps between W*-algebras

- P $f : A \to B$ is **positive** if it preserves positive elements
- sU f is **sub-unital** if $0 \leq f(1) \leq 1$ holds
- cP f is **completely positive** if for every $n \in \mathbb{N}$, $\mathcal{M}_n(f) : \mathcal{M}_n(A) \to \mathcal{M}_n(B)$ defined by $\mathcal{M}_n(f)([x_{i,j}]_{i,j\leq n}) = [f(x_{i,j})]_{i,j\leq n}$ is positive.
  - $\mathcal{M}_n(A)$: W*-algebra generated by the set of n-by-n matrices whose entries are in A.
- N $\phi : A \to B$ is **normal** if ϕ is positive and its restriction $\phi : [0,1]_A \to [0,1]_B$ is Scott-continuous.
  - $[0,1]_A$, subset of positive elements below the unit.

Löwner order

Definition

$\mathbf{W}^*$: category of W*-algebras together with NsU-maps

Definition (Löwner partial order)

For positive maps $f, g : A \to B$ between W*-algebras A and B, we define pointwise the following partial order $\sqsubseteq$: $f \sqsubseteq g$ if and only if $g - f$ is positive.

Theorem

For W-algebras A and B, the poset $(\mathbf{W}^*(A, B), \sqsubseteq)$ is* **directed-complete**.

W*-algebras are order-enriched

Recall: a category whose hom-sets are posets is called **$\mathbf{Dcpo}_{\perp!}$-enriched** if:

1. its hom-sets are dcpos with bottom
2. pre-composition and post-composition of morphisms are strict and Scott-continuous.

Theorem

*The category $\mathbf{W}^*$ is a $\mathbf{Dcpo}_{\perp!}$-**enriched** category.*

Theorem (K. Cho, 2014; done independently)

$\mathbf{W}^*{}_{\mathrm{cP}}$, *category of W*-algebras together with NcPsU-maps is* $\mathbf{Dcpo}_{\perp!}$*-enriched with the following order on maps: $f \sqsubseteq_{cP} g$ if and only if $g - f$ is completely positive.*

Von Neumann functors

Definition

An endofunctor F on a $\mathbf{Dcpo}_{\perp!}$-enriched category $\mathbf{C}$ is **locally continuous** if $F_{X,Y} : \mathbf{C}(X,Y) \to \mathbf{C}(FX,FY)$ is Scott-continuous.

Definition

A **von Neumann functor** is a locally continuous endofunctor on $\mathbf{W}^*$ which preserves multiplication-preserving maps.

Theorem

The category $\mathbf{W}^*$ *is* **algebraically compact for the class of von Neumann functors**, *i.e. every von Neumann functor F admits a canonical fixpoint and there is an isomorphism between the initial F-algebra and the inverse of the final F-coalgebra.*

Recipe: how to construct a fixpoint for such functors

- Consider a sequence of the form $\Delta = D_0 \xrightarrow{\alpha_0} D_1 \xrightarrow{\alpha_1} \cdots$ where $D_0 = 0$, $D_{n+1} = FD_n$, $\alpha_0 =!_{F0}$, $\alpha_{n+1} = F\alpha_n$ $(n \in \mathbb{N})$
- Define a W*-algebra D and turn it into a cocone $\mu : \Delta \to D$, i.e. a sequence of arrows $\mu_n : D_n \to D$ such that the equality $\mu_n = \mu_{n+1} \circ \alpha_n$ holds for every $n \geq 0$. This is a colimit of Δ
- Observe that $F\mu : F\Delta \to FD$ is a colimit for $F\Delta$, obtained by removing the first arrow from Δ.
- Two colimiting cocone with same vertices are isomorphic, which implies that D and FD share the same limit and are isomorphic.
- Dually, consider the sequence $\Delta^{\mathsf{op}} = D_0 \xleftarrow{\beta_0} D_1 \leftarrow \cdots$ and provide a limit for it.
- Conclusion: The functor F admits a fixpoint.

Examples

Functor	Fixpoint	Correspondence in **Sets**
$FX = X \oplus \mathbb{C}$	$\bigoplus_{i \geq 0} \mathbb{C} = \ell^\infty(\mathbb{N})$	Lifting
$FX = (X \otimes A) \oplus \mathbb{C}$	$\bigoplus_{i \geq 0} i \cdot A$	A^*

$[\![nat]\!] = \ell^\infty(\mathbb{N})$ We have **infinite** types!

Quantum types, revisited

$A, B := I \mid A \otimes B \mid 0 \mid A + B \mid bit \mid qbit \mid nat \mid \mu x.A$

- $[\![bit]\!] = \mathbb{C}^2 = \ell^\infty(2)$, $[\![trit]\!] = \mathbb{C}^3 = \ell^\infty(3)$, ...
- $[\![qbit]\!] = M_2(\mathbb{C}), [\![qtrit]\!] = M_3(\mathbb{C}), ...$
- $[\![0]\!] = 0 = \{0\}$
- $[\![I]\!] = \mathbb{C}$
- $[\![A + B]\!] = [\![A]\!] \oplus [\![B]\!]$
- $[\![A \otimes B]\!] = [\![A]\!] \otimes [\![B]\!]$
- $[\![nat]\!] = \ell^\infty(\mathbb{N})$
- $[\![\mu x.A]\!]$ = initial algebra of $[\![x \vdash A]\!]$
  = final coalgebra of $[\![x \vdash A]\!]$

Conclusion

- The category $\mathbf{W}^*$ of W*-algebras together with normal sub-unital maps is **order-enriched**.
- The category $\mathbf{W}^*$ is **algebraically compact** for an important class of functors.

This is the appropriate category for further investigations of the semantics of quantum computation.

This is a new trend that we are now working on at Radboud University.

Thank you