the coinductive resumption monad

maciej piróg and jeremy gibbons / u.oxford

mfpsxxx / ithaca, ny / june 2014

ingredient 1: resumptions

constructions to interpret processes

$\nu X.(O \times X)^I$ [Milner, Plotkin, Abramsky]

$RA = \mu X.M(FX + A)$ [Moggi]

$R'A = \nu X.M(FX + A)$ [Goncharov & Schröder's, Piróg & Gibbons]

ingredient 2: modules over monads

M — i'm a monad

S — i'm a (right) M-module, that is, an endofunctor such that there exists

$$\overline{\mu} : SM \to S$$

coherent with the structure of M

ingredient 3: idealised monads

$\langle M, \mu \rangle$ — i'm a monad and my own module

$\langle S, \overline{\mu}^S \rangle \to \langle G, \overline{\mu}^G \rangle$ — morphisms between M-modules are coherent morphisms $S \to G$

M is idealised with an M-module S if there exists a module morphism $\sigma : S \to M$.

ingredient 4: completely iterative monads

M — i'm idealised with $\langle \overline{M},\ \overline{\mu} : \overline{M}M \to \overline{M} \rangle$

$$e : X \to M(X + A)$$

$$\overbrace{\hphantom{xxxxxxxxxxxxxxxxxxxxxxxxxxxx}}$$

i'm an equation morphism

i'm guarded if i factor as:

$$X \to \overline{M}(X + A) + A \xrightarrow{[\sigma, \eta \cdot \mathsf{inr}]} M(X + A)$$

for a guarded morphism

$$e : X \to M(X + A)$$

there exists unique $e^\dagger : X \to MA$ such that

$$\begin{array}{ccc} X & \xrightarrow{\quad e^\dagger \quad} & MA \\ {\scriptstyle e}\downarrow & & \uparrow{\scriptstyle \mu_A} \\ M(X + A) & \xrightarrow{\quad M[e^\dagger, \eta_A] \quad} & M^2 A \end{array}$$

we define a morphism between two cims

M w.r.t. $\langle \overline{M}, \overline{\mu}^M \rangle$ and T w.r.t. $\langle \overline{T}, \overline{\mu}^T \rangle$

a monad morphism $m : M \to T$ s.t.
there exists

$$\overline{m} : \overline{M} \to \overline{T}$$

with $m \cdot \sigma^M = \sigma^T \cdot \overline{m} : \overline{M} \to T$

F — i'm an endofunctor!

$$F^\infty A = \nu X. FX + A$$

i'm a monad!

i'm a cim w.r.t. FF^∞!

i'm the free cim!

[Aczel & Adámek & Milius & Velebil]

preparation

M — i'm a completely iterative monad!

S – i'm an M-module!

$$MS^{\infty}$$

for example:

$$\nu X.M(FX + A) \cong M(\nu X.FMX + A) \cong M(FM)^{\infty}$$

some **results**

M — i'm a completely iterative monad!

S – i'm an M-module!

$$\underbrace{MS^{\infty}}_{\color{red}{\text{i'm a monad!}}}$$

monadic structure — the construction

M has a lifting $\langle\!\langle M \rangle\!\rangle$ in the category of Eilenberg-Moore algebras of S^∞, aka the category of Elgot algebras.

hence, there is a distributive law $\lambda : S^\infty M \to MS^\infty$.

hence, there is a monadic structure on MS^∞.

example a

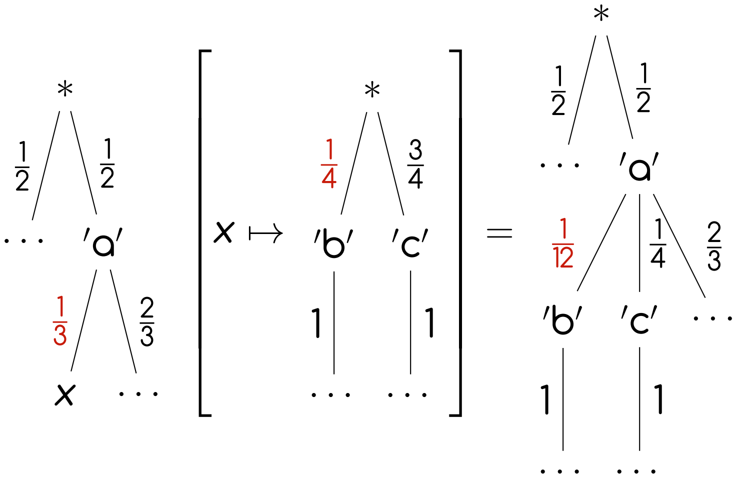
$\mathcal{D}$ — i'm the probability distribution monad

$O = \{'a', 'b', \ldots\}$ — i'm an alphabet

$O \times \mathcal{D}$ — i'm a $\mathcal{D}$-module

$\mathcal{D}(O \times \mathcal{D})^{\infty} X$ — i model a probabilistic process with variables from X that outputs possibly infinite words

example a continues



example b

A — i'm an object of possible states

$\mathcal{R}X = X^A$ — i'm a reader monad

$\mathcal{W}X = X \times A$ — i'm reader's module

$\mathcal{R}\mathcal{W}^\infty X = ((\text{-} \times A)^\infty X)^A$ — i'm a "logging" state monad

M — i'm a completely iterative monad!

S – i'm an M-module!

$$\overbrace{MS^{\infty}}^{\text{i'm a monad!}}$$

i'm a completely iterative monad!

$$MS^{\infty}$$

is a cim with respect to

$$\overline{M}S^{\infty} + MSS^{\infty}$$

back to example a

$e : X \to \mathcal{D}(O \times \mathcal{D})^{\infty}(X + A)$ — transition system

$\rho : 1 \to X$ — initial state

$e^{\dagger} \cdot \rho : 1 \to \mathcal{D}(O \times \mathcal{D})^{\infty} A$

the pessimistic module

$\mathcal{D}'X = \mathcal{D}(X+1)$ — i'm the probability distribution monad with failure and a cim with respect to 'at least 50% failure'

$$\mathcal{D}'(O \times \mathcal{D}')^{\infty} X$$

M — i'm a completely iterative monad!

S – i'm an M-module!

$$\underbrace{MS^{\infty}}_{\text{i'm a monad!}}$$

i'm a completely iterative monad!

i instantiate to a coproduct

$$M(FM)^*$$

is a coproduct of M and F^* in the category of monads

[Hyland & Plotkin & Power]

it turns out that

$$M(FM)^{\infty}$$

is a coproduct of M and F^{∞} in the category of completely iterative monads

actually...

$M(FM)^\infty$ is the coproduct of M and F^∞ in the category of monads completely iterative w.r.t. two-sided ideals.

but don't worry...

this category is a full reflective subcategory of the original one with an identity-on-the-monadic-structure left adjoint.

dessert: future work

semantics of processes along the lines of Abramsky *et al.*'s interaction categories?

least- vs greatest fixed points?

finitary case?

M — i'm a completely iterative monad!

S – i'm an M-module!

$$\underbrace{MS^{\infty}}_{\substack{\text{i'm a monad!} \\ \text{i'm a cim!} \\ \text{am i the free cim?}}}$$

the end

thank you for your attention / do read the paper / do ask me some questions