Labelled Markov Processes

A tutorial overview

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- Dexter Kozen, Kim Larsen, Radu Mardare

Summary of Results

- Probabilistic bisimulation can be defined for continuous state-space systems. [LICS97]
- Logical characterization. [LICS98,Info and Comp 2002]
- Metrics. [CONCUR99, TCS2004, UAI 2004, UAI 2005, SIAM J. Comp. 2011, QEST 2012]
- Approximation of LMPs. [LICS00,Info and Comp 2003, QEST 2005]
- Weak bisimulation. [LICS02,CONCUR02]
- Real time. [QEST 2004, JLAP 2003, LMCS 2006]
- Event bisimulation [CMCS 2004, Info and Comp 2006]
- Duality [LICS 2013, MFCS 2013, MFPS 2014]
- Approximation by averaging [CONCUR 2003, ICALP 2009, JACM 2014]
- Logic and approximation [MFCS 2012]

Definition

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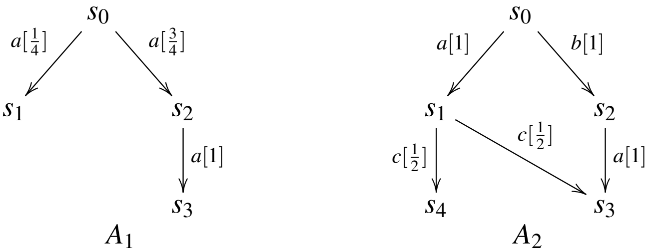
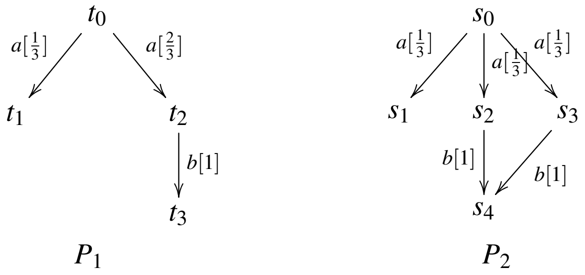
- $$(S, \mathsf{L}, \forall a \in \mathsf{L}\ T_a : S \times S \longrightarrow [0, 1])$$
- The model is reactive: All probabilistic data is internal - no probabilities associated with environment behaviour.

Examples of PTSs



Bisimulation for PTS: Larsen and Skou

- Consider



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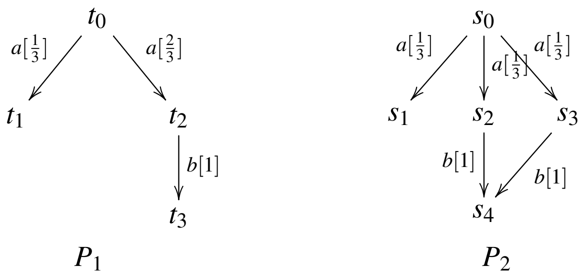

- Should s_0 and t_0 be bisimilar?
- Yes, but we need to add the probabilities.

The Official Definition

- Let $\mathcal{S} = (S, \mathsf{L}, T_a)$ be a PTS. An equivalence relation R on S is a **bisimulation** if whenever sRs', with $s, s' \in S$, we have that for all $a \in \mathcal{A}$ and every R-equivalence class, A, $T_a(s, A) = T_a(s', A)$.
- The notation $T_a(s, A)$ means "the probability of starting from s and jumping to a state in the set A."
- Two states are bisimilar if there is some bisimulation relation R relating them.

What are labelled Markov processes?

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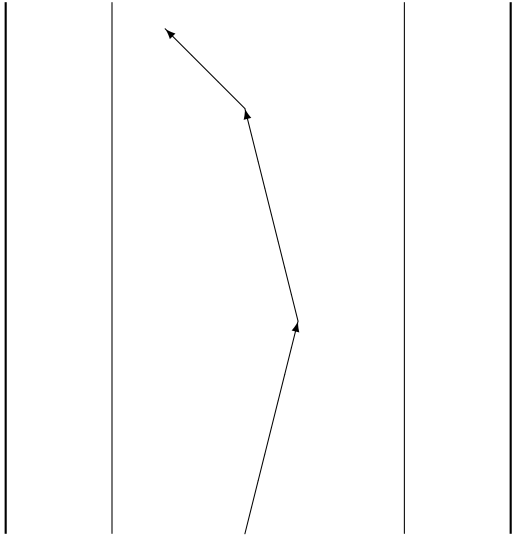
- All probabilistic data is internal - no probabilities associated with environment behaviour.
- We observe the interactions - not the internal states.
- **In general, the state space of a labelled Markov process may be a continuum.**

Motivation

Model and reason about systems with continuous state spaces or continuous time evolution or both.

- hybrid control systems; e.g. flight management systems.
- telecommunication systems with spatial variation; e.g. cell phones
- performance modelling,
- continuous time systems,
- probabilistic process algebra with recursion.

An Example of a Continuous-State System



a - turn left

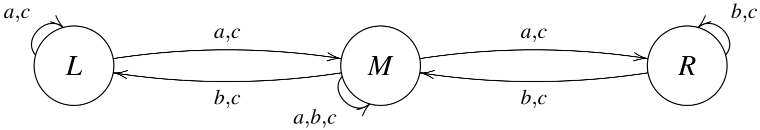
b - turn right

c - straight

Actions

a - turn left, b - turn right, c - keep on course
The actions move the craft sideways with some probability distributions on how far it moves. The craft may “drift” even with c. The action a (b) must be disabled when the craft is too near the left (right) boundary.

Schematic of Example

- 

- This picture is misleading: unless very special conditions hold the process cannot be compressed into an equivalent (?) finite-state model. In general, the transition probabilities should depend on the position.

Stochastic Kernels

- A stochastic kernel (Markov kernel) is a function $h : S \times \Sigma \longrightarrow [0, 1]$ with (a) $h(s, \cdot) : \Sigma \longrightarrow [0, 1]$ a (sub)probability measure and (b) $h(\cdot, A) : X \longrightarrow [0, 1]$ a measurable function.

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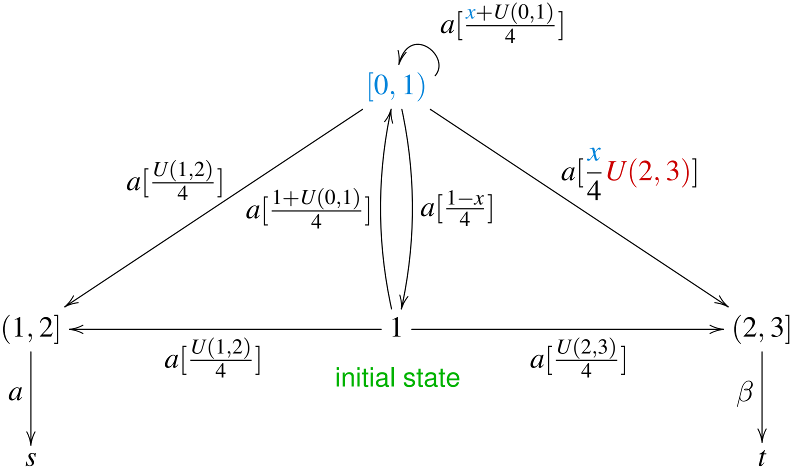
- Though apparantly asymmetric, these are the stochastic analogues of binary relations
- and the uncountable generalization of a matrix.

Formal Definition of LMPs

- An LMP is a tuple $(S, \Sigma, \mathsf{L}, \forall\alpha \in \mathsf{L}.\tau_\alpha)$ where $\tau_\alpha : S \times \Sigma \longrightarrow [0,1]$ is a transition probability function such that
- $\forall s : S.\lambda A : \Sigma.\tau_\alpha(s, A)$ is a subprobability measure
  and
  $\forall A : \Sigma.\lambda s : S.\tau_\alpha(s, A)$ is a measurable function.

Example of LMP

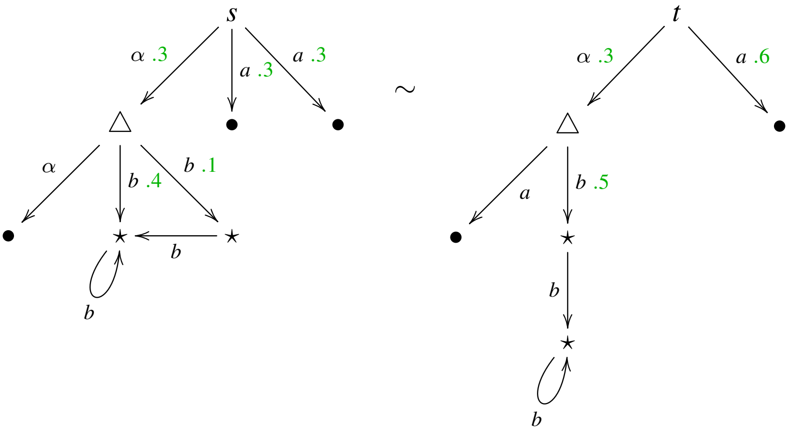


For $x \in [0,1)$, $\tau_a(x,[2.1,2.4]) = \frac{x}{4}0.3$

Larsen-Skou Bisimulation

- Let $\mathcal{S} = (S, i, \Sigma, \tau)$ be a labelled Markov process. An equivalence relation R on S is a **bisimulation** if whenever sRs', with $s, s' \in S$, we have that for all $a \in \mathcal{A}$ and every R-closed measurable set $A \in \Sigma$, $\tau_a(s, A) = \tau_a(s', A)$.
  Two states are bisimilar if they are related by a bisimulation relation.
- Can be extended to bisimulation between two different **LMP**s.

Larsen-Skou Bisimulation - Example



Logical Characterization

- $$\mathcal{L} ::== \mathsf{T} | \phi_1 \wedge \phi_2 | \langle a \rangle_q \phi$$
- We say $s \models \langle a \rangle_q \phi$ iff
$$\exists A \in \Sigma.(\forall s' \in A. s' \models \phi) \wedge (\tau_a(s, A) > q).$$

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$$\exists A \in \Sigma.(\forall s' \in A. s' \models \phi) \wedge (\tau_a(s, A) > q).$$
- Two systems are bisimilar iff they obey the same formulas of $\mathcal{L}$. [DEP 1998 LICS, I and C 2002]

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Event Bisimulation

Given a LMP (X, Σ, τ_a), an **event-bisimulation** is a sub-σ-algebra Λ of Σ such that (X, Λ, τ_a) is still an LMP.

Process Equivalence is Fundamental

- Markov chains:
- Lumpability
- Labelled Markov processes: Bisimulation
- Markov decision processes: Bisimulation
- Labelled Concurrent Markov Chains with τ transitions: Weak Bisimulation

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- In the context of probability is exact equivalence reasonable?
- We say “no”. A small change in the probability distributions may result in bisimilar processes no longer being bisimilar though they may be very “close” in behaviour.
- Instead one should have a (pseudo)metric for probabilistic processes.

A metric-based approximate viewpoint

- Move from equality between processes to distances between processes (Jou and Smolka 1990).

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- Move from equality between processes to distances between processes (Jou and Smolka 1990).
- Formalize distance as a metric:

$$d(s,s) = 0, d(s,t) = d(t,s), d(s,u) \leq d(s,t) + d(t,u).$$

Quantitative analogue of an equivalence relation.

Summary of results

- Establishing closeness of states: Coinduction

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- Compositional reasoning by *Non-Expansivity*.
  Process-combinators take nearby processes to nearby processes.

$$\frac{d(s_1, t_1) < \epsilon_1, \qquad d(s_2, t_2) < \epsilon_2}{d(s_1 \mid\mid s_2, t_1 \mid\mid t_2) < \epsilon_1 + \epsilon_2}$$

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- Results work for Markov chains, Labelled Markov processes, Markov decision processes and Labelled Concurrent Markov chains with τ-transitions.

Criteria on Metrics

- Soundness:

$$d(s,t) = 0 \Leftrightarrow \; s,t \text{ are bisimilar}$$

- Stability of distance under temporal evolution:“Nearby states stay close forever.”
- Metrics should be computable (efficiently?).

Bisimulation Recalled

Let R be an equivalence relation. R is a bisimulation if: $s\ R\ t$ if:

$$(s \longrightarrow P) \Rightarrow [t \longrightarrow Q, P =_R Q]$$

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where $P =_R Q$ if

$$(\forall R - \text{closed } E)\ P(E) = Q(E)$$

A putative definition of a metric-bisimulation

- m is a metric-bisimulation if: $m(s,t) < \epsilon \Rightarrow$:

$$s \longrightarrow P \Rightarrow \; t \longrightarrow Q, \quad m(P,Q) < \epsilon$$

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- Need a way to lift distances from states to a distances on distributions of states.

A detour: Kantorovich metric

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- $$d(\mu, \nu) = \sup_f |\int f d\mu - \int f d\nu|, f \text{ 1-Lipschitz}$$

A detour: Kantorovich metric

- Metrics on probability measures on metric spaces.
- $\mathcal{M}$: 1-bounded pseudometrics on states.
- $$d(\mu, \nu) = \sup_f |\int f d\mu - \int f d\nu|, f \text{ 1-Lipschitz}$$
- Arises in the solution of an LP problem: transshipment.

An LP version for Finite-State Spaces

When state space is finite: Let P, Q be probability distributions. Then:

$$m(P,Q) = \max \sum_i (P(s_i) - Q(s_i))a_i$$

subject to:

$$\forall i. 0 \leq a_i \leq 1$$
$$\forall i,j.\ a_i - a_j \leq m(s_i, s_j).$$

The Dual Form

- Dual form from Worrell and van Breugel:
- $$\min \sum_{i,j} l_{ij} m(s_i, s_j) + \sum_i x_i + \sum_j y_j$$

  subject to:

  $$\begin{array}{l} \forall i.\ \sum_j l_{ij} + x_i = P(s_i) \\ \forall j.\ \sum_i l_{ij} + y_j = Q(s_j) \\ \forall i,j.\ l_{ij}, x_i, y_j \geq 0. \end{array}$$
- We prove many equations by using the primal form to show one direction and the dual to show the other.

Return from Detour

Summary of detour: Given a metric on states in a metric space, can lift to a metric on probability distributions on states.

Metric "Bisimulation"

- m is a metric-bisimulation if: $m(s,t) < \epsilon \Rightarrow$:

$$s \longrightarrow P \Rightarrow \; t \longrightarrow Q, \quad m(P,Q) < \epsilon$$

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- The required canonical metric on processes is the least such: ie. the distances are the least possible.
- Thm: Canonical least metric exists. Usual fixed-point theory arguments.

Metrics: some details

- $\mathcal{M}$: 1-bounded pseudometrics on states with ordering

$$m_1 \preceq m_2 \text{ if } (\forall s,t)\ [m_1(s,t) \geq m_2(s,t)]$$

- $(\mathcal{M}, \preceq)$ is a complete lattice.
-

$$\begin{aligned}
\bot(s,t) &= \begin{cases} 0 \text{ if } s = t \\ 1 \text{ otherwise} \end{cases} \\
\top(s,t) &= 0, (\forall s,t) \\
(\sqcap\{m_i\}(s,t) &= \sup_i m_i(s,t)
\end{aligned}$$

Maximum fixed point definition

- Let $m \in \mathcal{M}$. $F(m)(s,t) < \epsilon$ if:

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- F is monotone on $\mathcal{M}$, and metric-bisimulation is the greatest fixed point of F.
- The closure ordinal of F is ω.

A logical metric

- Develop a real-valued “modal logic” based on the analogy due to Kozen:

Program Logic	Probabilistic Logic
State s	Distribution μ
Formula ϕ	Random Variable f
Satisfaction $s \models \phi$	$\int f \mathrm{d}\mu$

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- We did this before the LP based techniques became available.

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  where h 1-Lipschitz : $[0,1] \to [0,1]$ and $\gamma \in (0,1]$.

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- $d(s,t) = \sup_f |f(s) - f(t)|$
- Thm: d coincides with the canonical metric-bisimulation.

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- For $\gamma < 1$ there is an LP-based strongly-polynomial (in the number of constraints, and the number of bits of precision required) algorithm to compute the metric.
- For $\gamma = 1$ an algorithm to compute the metric has been discovered by van Breugel et al.

Approximation Results

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- For any LMP, we explicitly provide a (countable) sequence of approximants to it such that:
    1. For every logical property satisfied by a process, there is an element of the chain that also satisfies the property.
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  1. For every logical property satisfied by a process, there is an element of the chain that also satisfies the property.
  2. The sequence of approximants converges, in the metric defined before, to the process that is being approximated.
- The essential idea: approximate bisimulation.

Domain-theoretic approximation of LMPs

- we establish the following equivalence of categories:

$$\mathbf{LMP} \simeq \ \mathit{Proc}$$

where **LMP** is the category with objects **LMP**s and with morphisms simulations; and *Proc* is the solution to the recursive domain equation

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- The sequence of approximants is a directed set in the simulation ordering and the process being approximated is the sup of this directed set.

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- Approximation is formalized by “coarsening the σ-algebra” rather than by clustering points.

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- Functorial view of expectation values.
- Then bisimulation is naturally dualized and gives event bisimulation.
- Approximation is formalized by “coarsening the σ-algebra” rather than by clustering points.
- The approximations form a profinite family that gives the bisimulation-minimal version of the original LMP as a projective limit.

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