Labelled Markov Processes

A tutorial overview

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- Dexter Kozen, Kim Larsen, Radu Mardare
Probabilistic bisimulation can be defined for continuous state-space systems. [LICS97]
Logical characterization. [LICS98, Info and Comp 2002]
Approximation of LMPs. [LICS00, Info and Comp 2003, QEST 2005]
Weak bisimulation. [LICS02, CONCUR02]
Real time. [QEST 2004, JLAP 2003, LMCS 2006]
Event bisimulation [CMCS 2004, Info and Comp 2006]
Duality [LICS 2013, MFCS 2013, MFPS 2014]
Approximation by averaging [CONCUR 2003, ICALP 2009, JACM 2014]
Logic and approximation [MFCS 2012]
Definition

Just like a labelled transition system with probabilities associated with the transitions.
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\[(S, L, \forall a \in L \ T_a : S \times S \rightarrow [0, 1])\]
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\[ (S, L, \forall a \in L \ T_a : S \times S \rightarrow [0, 1]) \]

- The model is reactive: All probabilistic data is internal - no probabilities associated with environment behaviour.
Examples of PTSs
Bisimulation for PTS: Larsen and Skou

Consider

\[
\begin{align*}
P_1 & \quad t_0 \\
& \quad \downarrow \quad \downarrow \\
& \quad t_1 \quad t_2 \\
& \quad a[\frac{1}{3}] \quad a[\frac{2}{3}] \\
& \quad \downarrow \quad \downarrow \\
& \quad t_3 \\
& \quad b[1] \\
\end{align*}
\]

\[
\begin{align*}
P_2 & \quad s_0 \\
& \quad \downarrow \quad \downarrow \\
& \quad s_1 \quad s_2 \\
& \quad a[\frac{1}{3}] \quad a[\frac{1}{3}] \\
& \quad \downarrow \quad \downarrow \\
& \quad s_3 \quad s_4 \\
& \quad b[1] \quad b[1]
\end{align*}
\]

Should \( s_0 \) and \( t_0 \) be bisimilar?
Consider

Should $s_0$ and $t_0$ be bisimilar?

Yes, but we need to add the probabilities.
Let $S = (S, L, T_a)$ be a PTS. An equivalence relation $R$ on $S$ is a **bisimulation** if whenever $sR_s'$, with $s, s' \in S$, we have that for all $a \in A$ and every $R$-equivalence class, $A$, $T_a(s, A) = T_a(s', A)$.

The notation $T_a(s, A)$ means “the probability of starting from $s$ and jumping to a state in the set $A$.”

Two states are bisimilar if there is some bisimulation relation $R$ relating them.
What are labelled Markov processes?

Labelled Markov processes are probabilistic versions of labelled transition systems. Labelled transition systems where the final state is governed by a probability distribution - no other indeterminacy.
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- Labelled Markov processes are probabilistic versions of labelled transition systems. Labelled transition systems where the final state is governed by a probability distribution - no other indeterminacy.
- All probabilistic data is internal - no probabilities associated with environment behaviour.
- We observe the interactions - not the internal states.
- In general, the state space of a labelled Markov process may be a continuum.
Motivation

Model and reason about systems with continuous state spaces or continuous time evolution or both.

- hybrid control systems; e.g. flight management systems.
- telecommunication systems with spatial variation; e.g. cell phones
- performance modelling,
- continuous time systems,
- probabilistic process algebra with recursion.
An Example of a Continuous-State System

a - turn left

b - turn right

c - straight
Actions

\( a \) - turn left, \( b \) - turn right, \( c \) - keep on course

The actions move the craft sideways with some probability distributions on how far it moves. The craft may “drift” even with \( c \). The action \( a \) (\( b \)) must be disabled when the craft is too near the left (right) boundary.
This picture is misleading: unless very special conditions hold the process cannot be compressed into an equivalent (?) finite-state model. In general, the transition probabilities should depend on the position.
A stochastic kernel (Markov kernel) is a function $h : S \times \Sigma \rightarrow [0, 1]$ with (a) $h(s, \cdot) : \Sigma \rightarrow [0, 1]$ a (sub)probability measure and (b) $h(\cdot, A) : X \rightarrow [0, 1]$ a measurable function.
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Stochastic Kernels

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- Though apparently asymmetric, these are the stochastic analogues of binary relations.

- and the uncountable generalization of a matrix.
Formal Definition of LMPs

- An LMP is a tuple \((S, \Sigma, L, \forall \alpha \in L. \tau_\alpha)\) where \(\tau_\alpha : S \times \Sigma \rightarrow [0, 1]\) is a transition probability function such that
- \(\forall s : S. \lambda A : \Sigma. \tau_\alpha(s, A)\) is a subprobability measure and
- \(\forall A : \Sigma. \lambda s : S. \tau_\alpha(s, A)\) is a measurable function.
Example of LMP

For $x \in [0, 1)$, $\tau_a(x, [2.1, 2.4]) = \frac{x}{4} 0.3$
Larsen-Skou Bisimulation

Let $S = (S, i, \Sigma, \tau)$ be a labelled Markov process. An equivalence relation $R$ on $S$ is a **bisimulation** if whenever $sRs'$, with $s, s' \in S$, we have that for all $a \in A$ and every $R$-closed measurable set $A \in \Sigma$, $\tau_a(s, A) = \tau_a(s', A)$.

Two states are bisimilar if they are related by a bisimulation relation.

Can be extended to bisimulation between two different LMPs.
Larsen-Skou Bisimulation - Example
Logical Characterization

\[ \mathcal{L} ::= T | \phi_1 \land \phi_2 | \langle a \rangle_q \phi \]

We say \( s \models \langle a \rangle_q \phi \) iff

\[ \exists A \in \Sigma. (\forall s' \in A. s' \models \phi) \land (\tau_a(s, A) > q). \]
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Two systems are bisimilar iff they obey the same formulas of \( \mathcal{L} \).
[DEP 1998 LICS, I and C 2002]
In measure theory one should focus on measurable sets rather than on points.
Event bisimulation

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Event Bisimulation

Given a LMP \((X, \Sigma, \tau_a)\), an **event-bisimulation** is a sub-\(\sigma\)-algebra \(\Lambda\) of \(\Sigma\) such that \((X, \Lambda, \tau_a)\) is still an LMP.
Process Equivalence is Fundamental

- Markov chains:
- Lumpability
- Labelled Markov processes: Bisimulation
- Markov decision processes: Bisimulation
- Labelled Concurrent Markov Chains with $\tau$ transitions: Weak Bisimulation
But...

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- We say “no”. A small change in the probability distributions may result in bisimilar processes no longer being bisimilar though they may be very “close” in behaviour.
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We say “no”. A small change in the probability distributions may result in bisimilar processes no longer being bisimilar though they may be very “close” in behaviour.

Instead one should have a (pseudo)metric for probabilistic processes.
A metric-based approximate viewpoint

- Move from equality between processes to distances between processes (Jou and Smolka 1990).

Formalize distance as a metric:

\[ d(s, s) = 0, \quad d(s, t) = d(t, s), \quad d(s, u) \leq d(s, t) + d(t, u). \]

Quantitative analogue of an equivalence relation.
A metric-based approximate viewpoint

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Summary of results

- Establishing closeness of states: Coinduction
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- Distinguishing states: Real-valued modal logics
- Equational and logical views coincide: Metrics yield same distances as real-valued modal logics
- Compositional reasoning by *Non-Expansivity*. Process-combinators take nearby processes to nearby processes.

\[
d(s_1, t_1) < \epsilon_1, \quad d(s_2, t_2) < \epsilon_2 \\
\Rightarrow d(s_1 \parallel s_2, t_1 \parallel t_2) < \epsilon_1 + \epsilon_2
\]
Summary of results

- Establishing closeness of states: Coinduction
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- Compositional reasoning by *Non-Expansivity*. Process-combinators take nearby processes to nearby processes.

\[
d(s_1, t_1) < \epsilon_1, \quad d(s_2, t_2) < \epsilon_2
\]
\[
d(s_1 || s_2, t_1 || t_2) < \epsilon_1 + \epsilon_2
\]

- Results work for Markov chains, Labelled Markov processes, Markov decision processes and Labelled Concurrent Markov chains with \(\tau\)-transitions.
Criteria on Metrics

- Soundness:
  \[ d(s, t) = 0 \iff s, t \text{ are bisimilar} \]

- Stability of distance under temporal evolution: “Nearby states stay close forever.”

- Metrics should be computable (efficiently?).
Bisimulation Recalled

Let $R$ be an equivalence relation. $R$ is a bisimulation if: $s \sim R t$ if:

$$(s \xrightarrow{} P) \Rightarrow [t \xrightarrow{} Q, P =_R Q]$$

$$(t \xrightarrow{} Q) \Rightarrow [s \xrightarrow{} P, P =_R Q]$$

where $P =_R Q$ if

$$(\forall R - \text{closed } E) P(E) = Q(E)$$
A putative definition of a metric-bisimulation

- $m$ is a metric-bisimulation if: $m(s, t) < \epsilon \Rightarrow$

\[
\begin{align*}
    s &\quad \xrightarrow{} P \quad \Rightarrow \quad t \quad \xrightarrow{} Q, \quad m(P, Q) < \epsilon \\
    t &\quad \xrightarrow{} Q \quad \Rightarrow \quad s \quad \xrightarrow{} P, \quad m(P, Q) < \epsilon
\end{align*}
\]
A putative definition of a metric-bisimulation

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  \[
  s \rightarrow P \Rightarrow t \rightarrow Q, \quad m(P, Q) < \epsilon
  \]
  \[
  t \rightarrow Q \Rightarrow s \rightarrow P, \quad m(P, Q) < \epsilon
  \]

- Problem: what is $m(P, Q)$? — Type mismatch!!
A putative definition of a metric-bisimulation

- $m$ is a metric-bisimulation if: $m(s, t) < \epsilon \Rightarrow$
  
  $s \xrightarrow{P} t \xrightarrow{Q}, \quad m(P, Q) < \epsilon$
  
  $t \xrightarrow{Q} s \xrightarrow{P}, \quad m(P, Q) < \epsilon$

- Problem: what is $m(P, Q)$? — Type mismatch!!
- Need a way to lift distances from states to a distances on distributions of states.
A detour: Kantorovich metric

- Metrics on probability measures on metric spaces.
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- $\mathcal{M}$: 1-bounded pseudometrics on states.
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$$d(\mu, \nu) = \sup_f | \int f d\mu - \int f d\nu |, f \text{ 1-Lipschitz}$$
A detour: Kantorovich metric

- Metrics on probability measures on metric spaces.
- $\mathcal{M}$: 1-bounded pseudometrics on states.

$$d(\mu, \nu) = \sup_f |\int f d\mu - \int f d\nu|, f \text{ 1-Lipschitz}$$

- Arises in the solution of an LP problem: transshipment.
An LP version for Finite-State Spaces

When state space is finite: Let $P, Q$ be probability distributions. Then:

$$m(P, Q) = \max \sum_i (P(s_i) - Q(s_i))a_i$$

subject to:

$$\forall i. 0 \leq a_i \leq 1$$
$$\forall i, j. a_i - a_j \leq m(s_i, s_j).$$
The Dual Form

- Dual form from Worrell and van Breugel:

\[
\min \sum_{i,j} l_{ij}m(s_i, s_j) + \sum_i x_i + \sum_j y_j
\]

subject to:

\[
\forall i. \sum_j l_{ij} + x_i = P(s_i) \\
\forall j. \sum_i l_{ij} + y_j = Q(s_j) \\
\forall i,j. l_{ij}, x_i, y_j \geq 0.
\]

- We prove many equations by using the primal form to show one direction and the dual to show the other.
Return from Detour

Summary of detour: Given a metric on states in a metric space, can lift to a metric on probability distributions on states.
Metric “Bisimulation”

- \( m \) is a metric-bisimulation if: \( m(s, t) < \epsilon \Rightarrow: \)

\[
s \xrightarrow{} P \Rightarrow t \xrightarrow{} Q, \quad m(P, Q) < \epsilon
\]

\[
t \xrightarrow{} Q \Rightarrow s \xrightarrow{} P, \quad m(P, Q) < \epsilon
\]

- The required canonical metric on processes is the least such: ie. the distances are the least possible.

- Thm: Canonical least metric exists. Usual fixed-point theory arguments.
Metrics: some details

- $\mathcal{M}$: 1-bounded pseudometrics on states with ordering

$$ m_1 \preceq m_2 \text{ if } (\forall s, t) \left[ m_1(s, t) \geq m_2(s, t) \right] $$

- $(\mathcal{M}, \preceq)$ is a complete lattice.

$$ \bot(s, t) = \begin{cases} 0 & \text{if } s = t \\ 1 & \text{otherwise} \end{cases} $$

$$ \top(s, t) = 0, (\forall s, t) $$

$$ (\sqcap \{m_i\})(s, t) = \sup_i m_i(s, t) $$
Maximum fixed point definition

Let \( m \in M \). \( F(m)(s, t) < \epsilon \) if:

\[
\begin{align*}
  s \rightarrow P & \Rightarrow t \rightarrow Q, \quad m(P, Q) < \epsilon \\
  t \rightarrow Q & \Rightarrow s \rightarrow P, \quad m(P, Q) < \epsilon
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Maximum fixed point definition

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- $F(m)(s, t)$ can be given by an explicit expression.
- $F$ is monotone on $\mathcal{M}$, and metric-bisimulation is the greatest fixed point of $F$. 
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- $F(m)(s, t)$ can be given by an explicit expression.
- $F$ is monotone on $M$, and metric-bisimulation is the greatest fixed point of $F$.
- The closure ordinal of $F$ is $\omega$. 
A logical metric

Develop a real-valued “modal logic” based on the analogy due to Kozen:

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- Define a metric based on how closely the random variables agree.
- We did this before the LP based techniques became available.
Real-valued Modal Logic

\[ f ::= 1 \mid \max(f, f) \mid h \circ f \mid \langle a \rangle f \]
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1(s) = 1 \quad \text{True}

\max(f_1, f_2)(s) = \max(f_1(s), f_2(s)) \quad \text{Conjunction}

h \circ f(s) = h(f(s)) \quad \text{Lipschitz}

\langle a \rangle f(s) = \gamma \int_{s' \in S} f(s') \tau_a(s, ds') \quad \text{a-transition}

where \( h \) 1-Lipschitz : \([0, 1] \rightarrow [0, 1]\) and \( \gamma \in (0, 1] \).
Real-valued Modal Logic

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\[ d(s, t) = \sup_f |f(s) - f(t)| \]
Real-valued Modal Logic

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Thm: \( d \) coincides with the canonical metric-bisimulation.
The role of $\gamma$

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- For $\gamma < 1$ there is an LP-based strongly-polynomial (in the number of constraints, and the number of bits of precision required) algorithm to compute the metric.
The role of $\gamma$

- $\gamma$ discounts the value of future steps.
- $\gamma < 1$ and $\gamma = 1$ yield very different topologies.
- For $\gamma < 1$ there is an LP-based strongly-polynomial (in the number of constraints, and the number of bits of precision required) algorithm to compute the metric.
- For $\gamma = 1$ an algorithm to compute the metric has been discovered by van Breugel et al.
Our main result is a systematic approximation scheme for labelled Markov processes. The set of LMPs is a Polish space.
Approximation Results

Our main result is a systematic approximation scheme for labelled Markov processes. The set of LMPs is a Polish space.

For any LMP, we explicitly provide a (countable) sequence of approximants to it such that:

1. For every logical property satisfied by a process, there is an element of the chain that also satisfies the property.
2. The sequence of approximants converges, in the metric defined before, to the process that is being approximated.
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1. For every logical property satisfied by a process, there is an element of the chain that also satisfies the property.
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The essential idea: approximate bisimulation.
Domain-theoretic approximation of LMPs

we establish the following equivalence of categories:

\[ \text{LMP} \simeq \text{Proc} \]

where \text{LMP} is the category with objects LMPs and with morphisms simulations; and \text{Proc} is the solution to the recursive domain equation

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- We show that there is a perfect match between:
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  - bisimulation and equality in \( \text{Proc} \),
  - simulation and the partial order of \( \text{Proc} \),
  - strict simulation and way below in \( \text{Proc} \).

- The sequence of approximants is a directed set in the simulation ordering and the process being approximated is the sup of this directed set.
Approximation by averaging

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- Functorial view of expectation values.
- Then bisimulation is naturally dualized and gives event bisimulation.

Approximation is formalized by “coarsening the $\sigma$-algebra” rather than by clustering points. The approximations form a profinite family that gives the bisimulation-minimal version of the original LMP as a projective limit.
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- Functorial view of expectation values.
- Then bisimulation is naturally dualized and gives event bisimulation.
- Approximation is formalized by “coarsening the $\sigma$-algebra” rather than by clustering points.
Approximation by averaging

- The latest idea is to view LMPs as function transformers.
- Functorial view of expectation values.
- Then bisimulation is naturally dualized and gives event bisimulation.
- Approximation is formalized by “coarsening the $\sigma$-algebra” rather than by clustering points.
- The approximations form a profinite family that gives the bisimulation-minimal version of the original LMP as a projective limit.
Conclusions

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