A Metrized Duality Theorem for Markov Processes

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Complex networks/systems are often modelled as stochastic processes

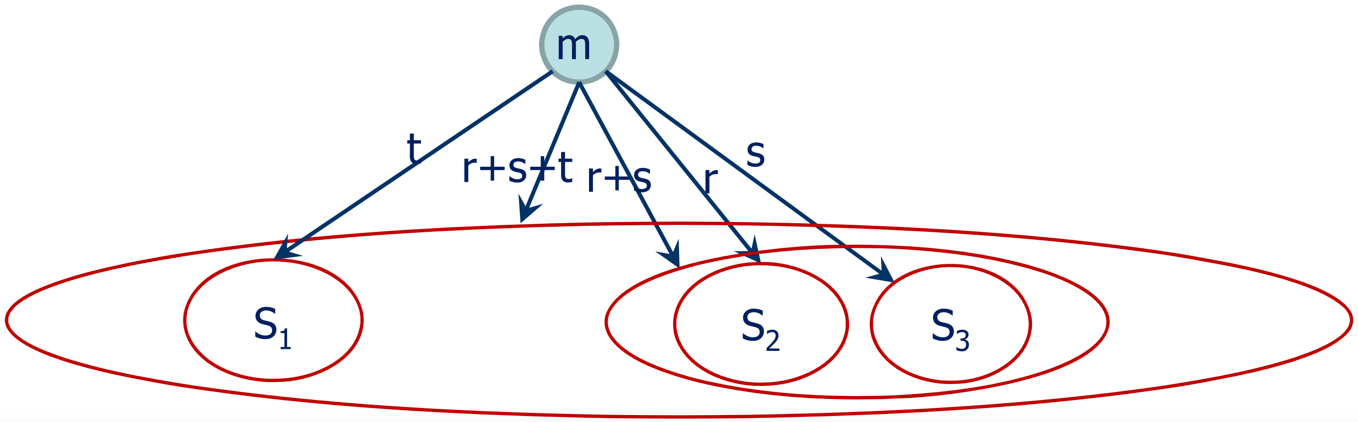
- to encapsulate a lack of knowledge or inherent non-determinism,
- to approximate the complex behaviour of real systems that cannot be modeled exactly since exact data are unknown

A **Markov process** is a measurable mapping

$$\theta : M \rightarrow \Delta(M,\Sigma)$$

where (M,Σ) is a measurable state space (analytic space),

and $\Delta(M,\Sigma)$ is the space of measures on (M,Σ).



Background

Stochastic/probabilistic/subprobabilistic Bisimulation

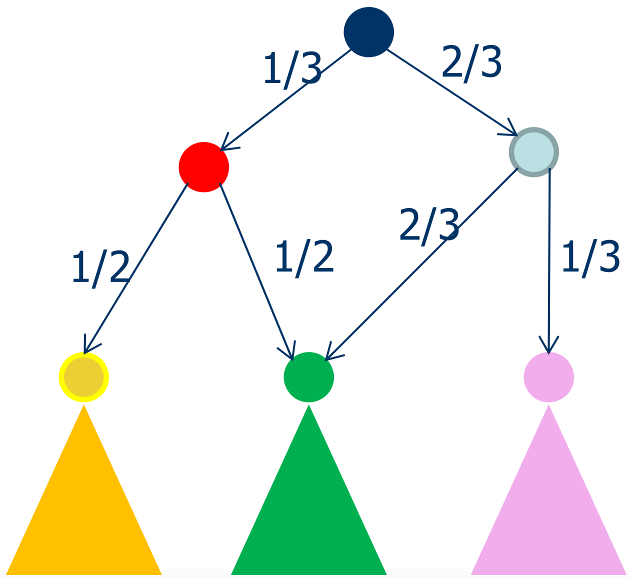
- equates systems with identical (probabilistic) behaviours

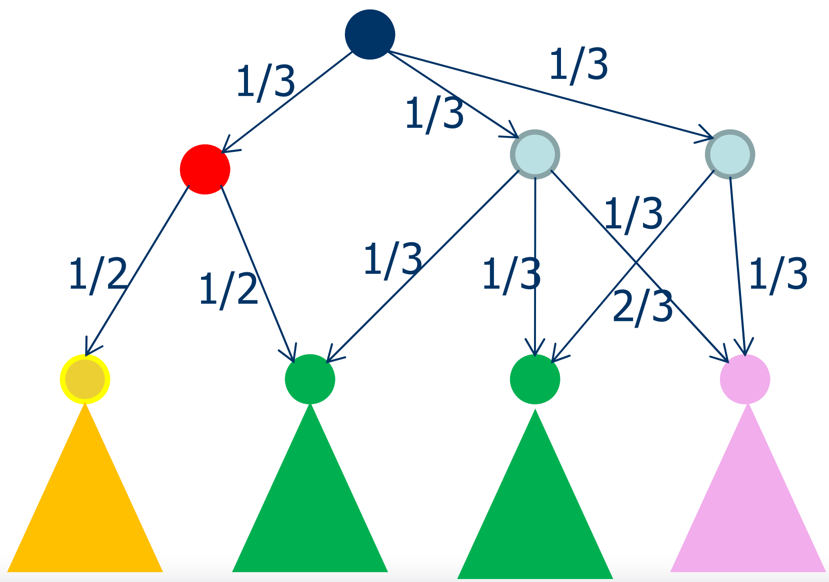
Given $\theta : M \rightarrow \Delta(M,\Sigma)$, a **bisimulation** is a relation (equivalence)

$$R \subseteq M \times M$$

s.t. mRn implies

- $\forall\ S \in \Sigma(R)\ ,\ \theta(m)(S)=\theta(n)(S)$





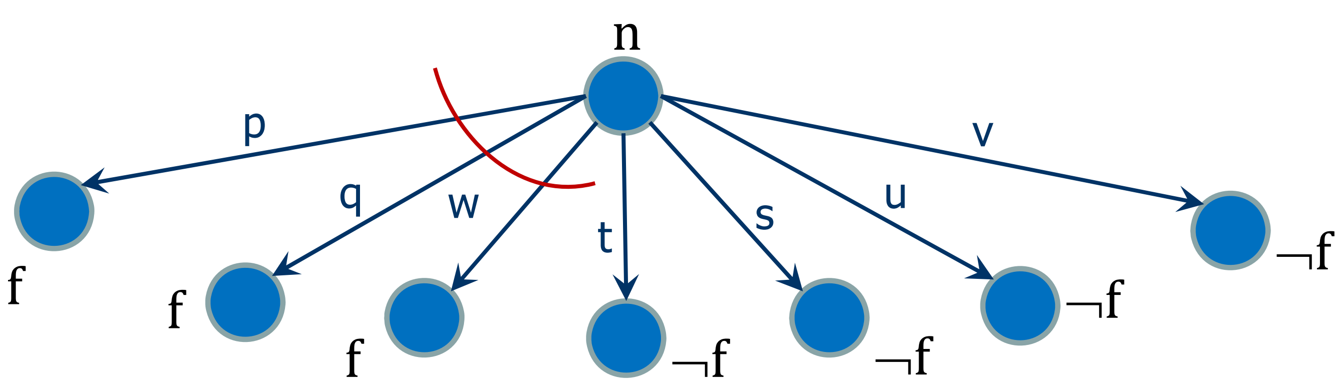
Markovian Logics

Syntax: $f := \bot \mid f \to f \mid L_r f \mid M_r f \qquad r \in \mathbb{Q}_+$

Semantics: Given $\theta : M \to \Delta(M,\Sigma)$, $m \in M$

$m \models L_r f$ iff $\theta(m)([f]) \geq r$ where $[f]=\{n \in M \mid n \models f\}$

$m \models M_r f$ iff $\theta(m)([f]) \leq r$



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Semantics: Given $\theta : M \rightarrow \Delta(M,\Sigma)$, $m \in M$

$m \vDash L_r f$ iff $\theta(m)([f]) \geq r$ where $[f] = \{n \in M \mid n \vDash f\}$

$m \vDash M_r f$ iff $\theta(m)([f]) \leq r$

Logical characterisation:

Given $\theta : M \rightarrow \Delta(M,\Sigma)$,

$$m \sim n \quad \text{iff} \quad [\forall f \in \mathcal{L},\ m \vDash f \ \text{iff}\ n \vDash f\ ].$$

Sound-Complete axiomatization: For any $F \subseteq \mathcal{L}$ and $f \in \mathcal{L}$,

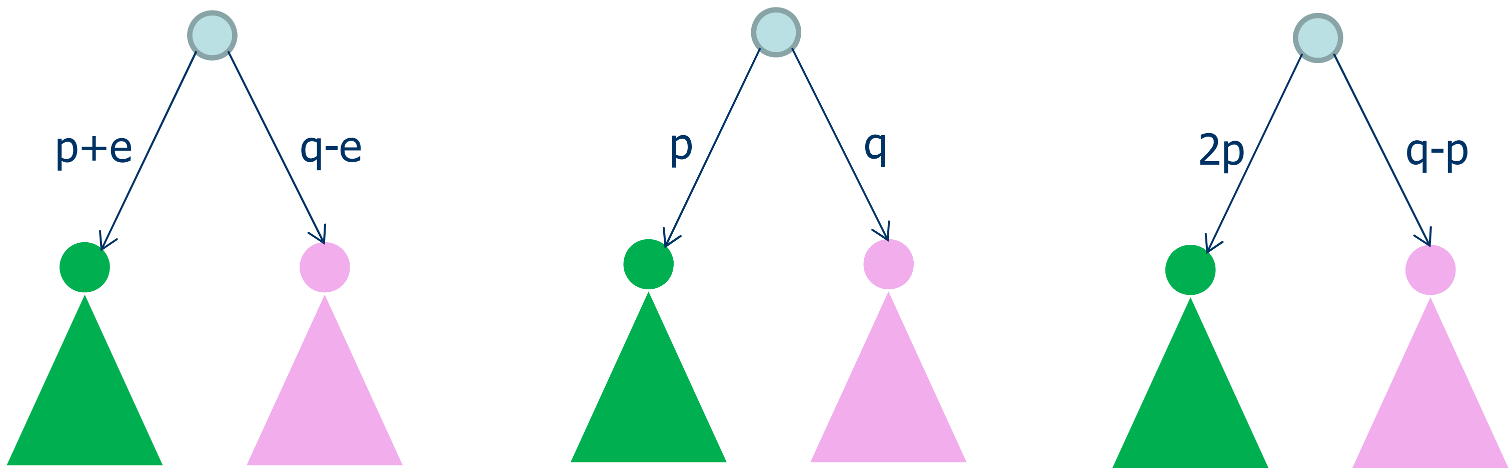
$$F \vDash f \ \text{iff}\ F \vdash f$$

Model construction using maximal consistent sets of formulas.

Practical perspective – the first argument

Bisimilarity is a too strict concept

- the interest is to understand when two systems have similar behaviours



Practical perspective – the first argument

Bisimilarity is a too strict concept

- the interest is to understand when two systems have <u>similar behaviours</u>

bisimilarity => bisimilarity distance (pseudometric)

$$d: M \times M \rightarrow [0,1]$$

P1. $d(m,n)=d(n,m)$

P2. $d(m,n) \leq d(m,m')+d(m',n)$

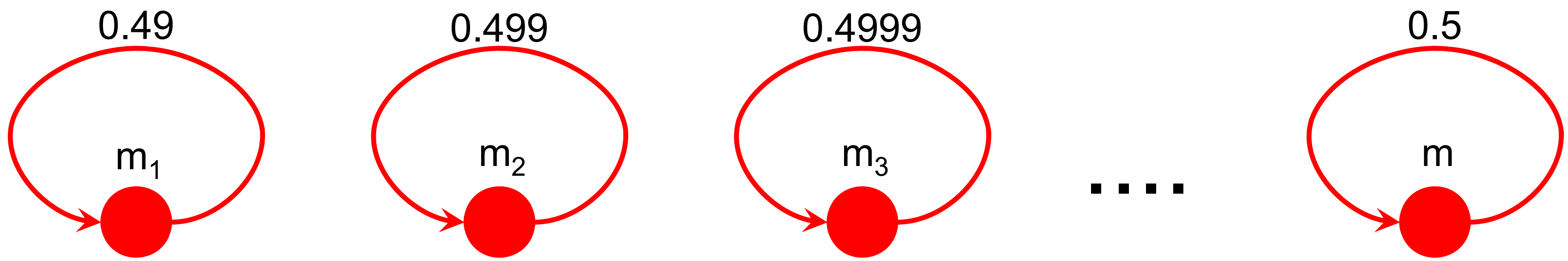
P3. $d(m,n)=0$ iff $m \sim n$

Practical perspective – the second argument

Often in science we

- approximate a real system
- check properties of better and better approximations
- extrapolate the results to the real system.

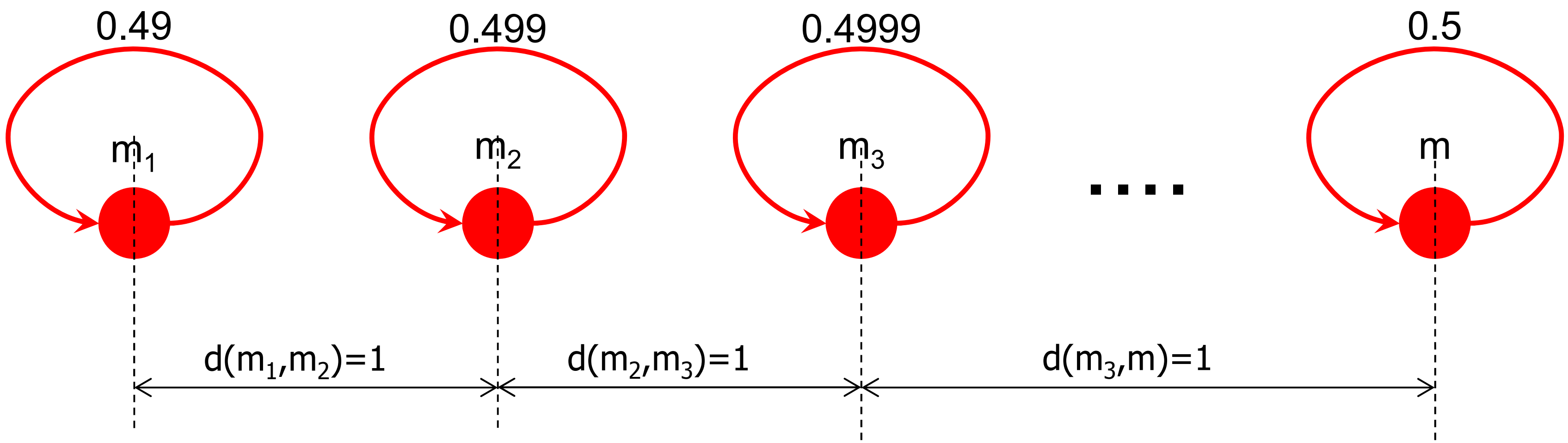
Assuming that we have a behavioural distance d, we implicitly assume some convergences



A proper bisimilarity distance must prove such a convergence in the open-ball topology!

An example of a *not so useful* bisimilarity distance

$$d(m,n) = \begin{cases} 0 & \text{if } m \sim n \\ 1 & \text{otherwise.} \end{cases}$$



The sequence is not Cauchy!

Practical perspective – the third argument

Often in science we

- approximate a real system
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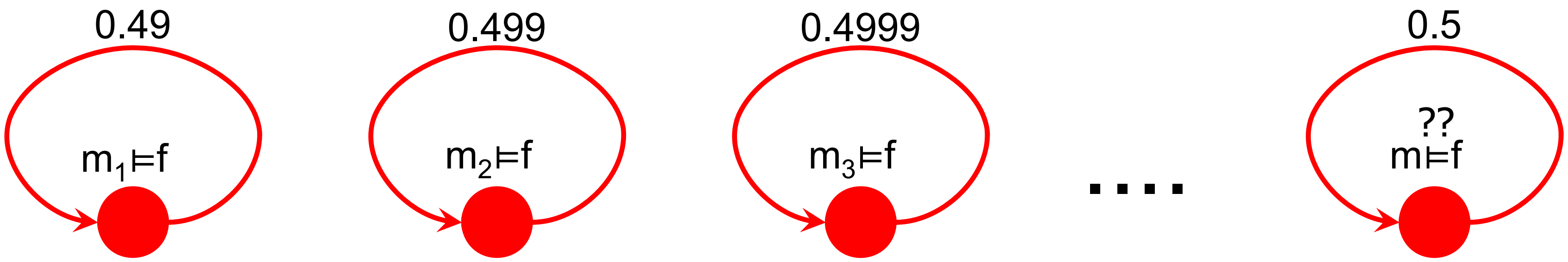
- extrapolate the results to the real system.

Assuming that we have a behavioural distance d, we implicitly assume that:

<u>Conjecture 1:</u> If $\lim m_k = m$ and for each k, $m_k \vDash f$, then $m \vDash f$.

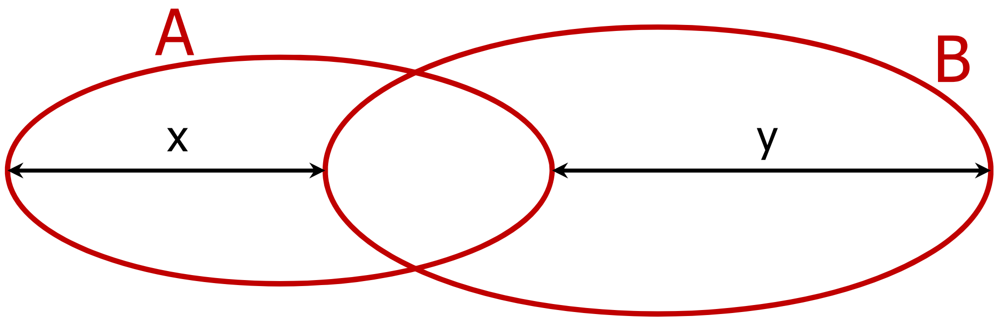


Practical perspective – the fourth argument

Any pseudometric $d: M \times M \to [0,1]$

induces a Hausdorff pseudometric $d^H : 2^M \times 2^M \to [0,1]$

$$d^H(A,B) = \max\{x,y\}$$



$$x = \sup_{a \in A} \inf_{b \in B} d(a,b) \qquad y = \sup_{b \in B} \inf_{a \in A} d(a,b)$$

Practical perspective – the fourth argument

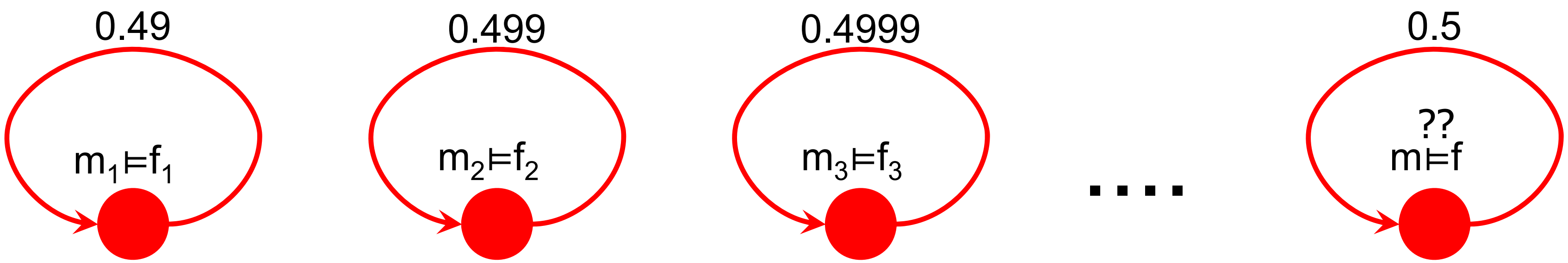
A logical property can be identified with the set of its models.
Hence, we get a pseudometric

$$d^H : \mathcal{L} \times \mathcal{L} \rightarrow [0,1]$$

and a topology over the space of formulas.

A possible convergence in $\mathcal{L}$:

$$L_{0.499..}T \xrightarrow{d^H} L_{0.5}T$$



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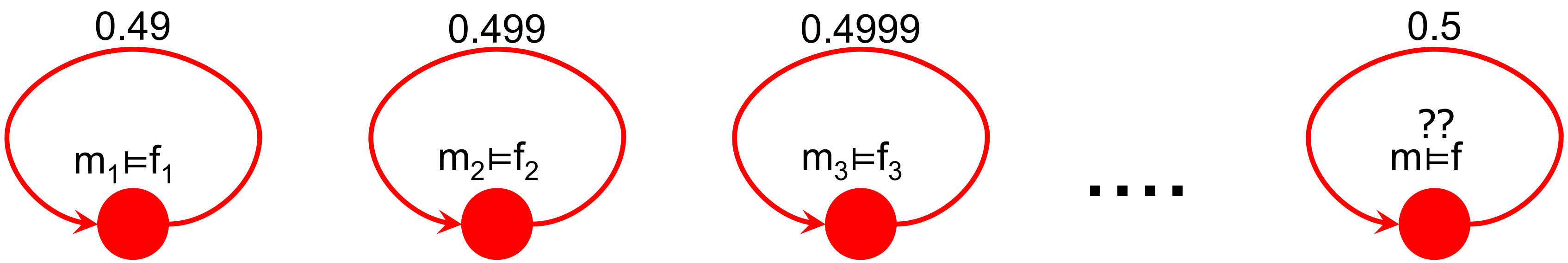
Conjecture 1: If $\lim m_k = m$ and for each k, $m_k \vDash f$, then $m \vDash f$.

Conjecture 2: If $\begin{cases} m_k \xrightarrow{d} m \\ f_k \xrightarrow{d^H} f \end{cases}$ and for each k, $m_k \vDash f_k$, then $m \vDash f$.



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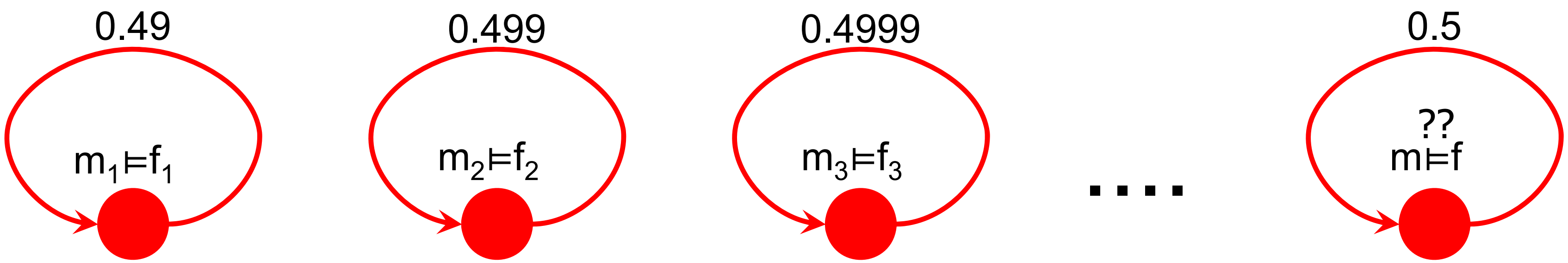
<u>Conjecture 1:</u> If $\lim m_k = m$ and for each k, $m_k \vDash f$, then $m \vDash f$. =>NO!

<u>Conjecture 2:</u> If $\begin{cases} m_k \xrightarrow{d} m \\ f_k \xrightarrow{d^H} f \end{cases}$ and for each k, $m_k \vDash f_k$, then $m \vDash f$. =>NO!



The topological space of logical formulas

The probabilistic case:

$\mathcal{L}$: $f ::= \bot \mid f \to f \mid L_r f$ $\quad r \in \mathbb{Q} \cap [0,1]$

$\mathcal{L}^+$ $g := T \mid g \wedge g \mid g \vee g \mid Lrf \mid M_r f$ $\quad f \in \mathcal{L}$

$\mathcal{L}^-$ $\{\neg g \mid g \in \mathcal{L}^+\}$

Proposition: Let d be a bisimilarity distance on probabilistic MPs.

1. If $f \in \mathcal{L}^+$, then [f] is a closed set in the open ball topology of d.
2. If $f \in \mathcal{L}^-$, then [f] is an open set in the open ball topology of d.

[MFCS 2012]

The topological space of logical formulas

The general (subprobabilistic, stochastic) case:

$\mathcal{L}$	$f := \bot \mid f \to f \mid L_r f \mid M_r f$	$r \in \mathbb{Q}_+$
$\mathcal{L}_0$	$f := \bot \mid f \to f \mid L_r f$	$r \in \mathbb{Q}_+$
$\mathcal{L}^+$	$g := \top \mid g \wedge g \mid g \vee g \mid L_r f \mid M_r f$	$f \in \mathcal{L}$
$\mathcal{L}_0^+$	$g := \top \mid g \wedge g \mid g \vee g \mid L_r f$	$f \in \mathcal{L}$

$\mathcal{L}^- := \{\neg g \mid g \in \mathcal{L}^+\}$, $\mathcal{L}_0^- := \{\neg g \mid g \in \mathcal{L}_0^+\}$

Proposition: Let d be a dynamically-continuous bisimilarity distance on DMPs.

1. If $f \in \mathcal{L}_0^+$, then $[f]$ is a closed set in the open ball topology induced by d.
2. If $f \in \mathcal{L}_0^-$, then $[f]$ is an open set in the open ball topology induced by d.
3. If $f \in \mathcal{L}^+$, then $[f]$ is a G_δ set (countable intersection of open sets).
4. If $f \in \mathcal{L}^-$, then $[f]$ is a F_σ set (countable union of closed sets).

[MFCS 2012]

Mathematical perspective

Can we characterize the behavioural distances that behave topologically correct?

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For $\theta : M \to \Delta(M,\Sigma)$, a **bisimulation** is a relation $R \subseteq M \times M$
s.t. mRn implies

- $\forall\, S \in \Sigma(R)$, $\theta(m)(S)=\theta(n)(S)$

A first attempt

For $\theta : M \to \Delta(M,\Sigma)$, a "good" bisimilarity distance is a pseudometric
$d: M \times M \to [0,1]$ such that for any sequence $(m_k)_k$ with $m_k \xrightarrow{d} m$

- $\forall\, S \in \Sigma(\sim)$, $\theta(m_k)(S) \xrightarrow{\mathbb{R}} \theta(m)(S)$

Can we characterize the behavioural distances that behave topologically correct?

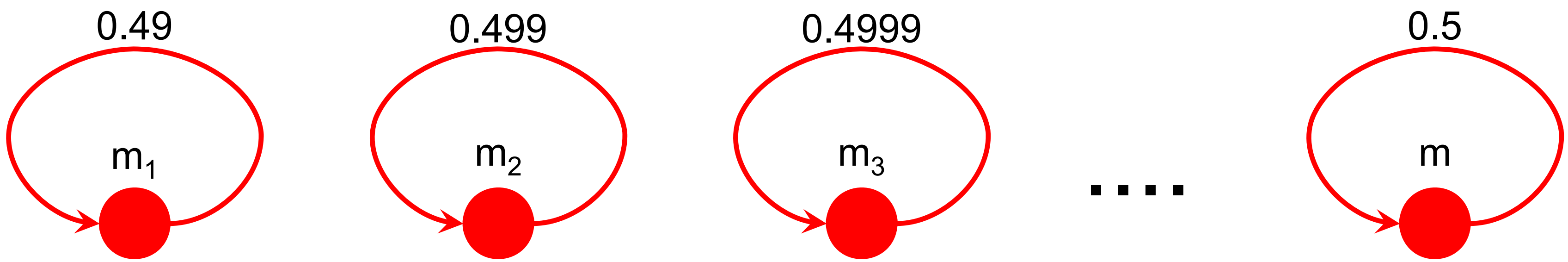
For $\theta : M \rightarrow \Delta(M,\Sigma)$, a **bisimulation** is a relation $R \subseteq M \times M$
s.t. mRn implies

- $\forall S \in \Sigma(R)$, $\theta(m)(S) = \theta(n)(S)$

A first attempt - Wrong! It misses the "<u>coinductive nature</u>"!

For $\theta : M \rightarrow \Delta(M,\Sigma)$, a "good" bisimilarity distance is a pseudometric $d: M \times M \rightarrow [0,1]$ such that for any sequence $(m_k)_k$ with $m_k \xrightarrow{d} m$

- $\forall S \in \Sigma(\sim)$, $\theta(m_k)(S) \xrightarrow{\mathbb{R}} \theta(m)(S)$



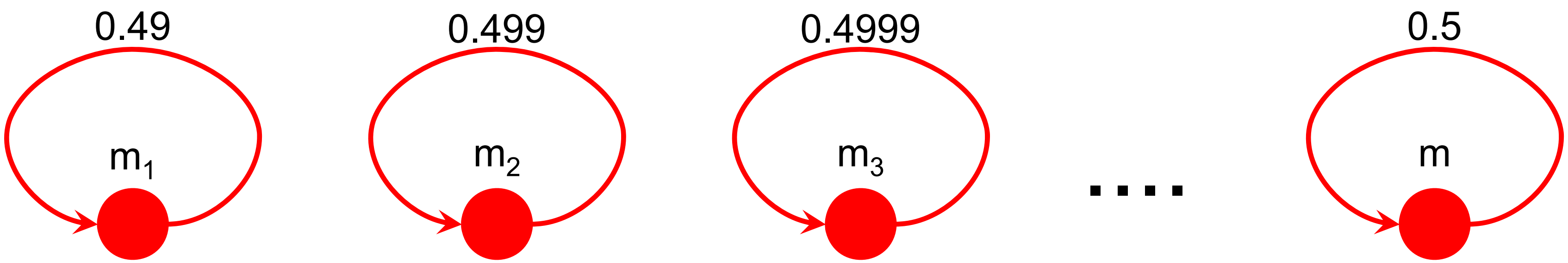
$\theta(m_k)(\{m\}) = 0$ for any k and $\theta(m)(\{m\}) = 0.5$

Mathematical perspective

A second attempt

For $\theta : M \to \Delta(M,\Sigma)$, a "good" bisimilarity distance is a pseudometric $d: M \times M \to [0,1]$ such that for any sequence $(m_k)_k$ with $m_k \xrightarrow{d} m$,

- $\forall S \in \Sigma(\sim)$, $\forall (S_k)_k \subseteq \Sigma(\sim)$ such that $S_k \xrightarrow{d^H} S$
  - $\theta(m_k)(S_k) \xrightarrow{\mathbb{R}} \theta(m)(S)$

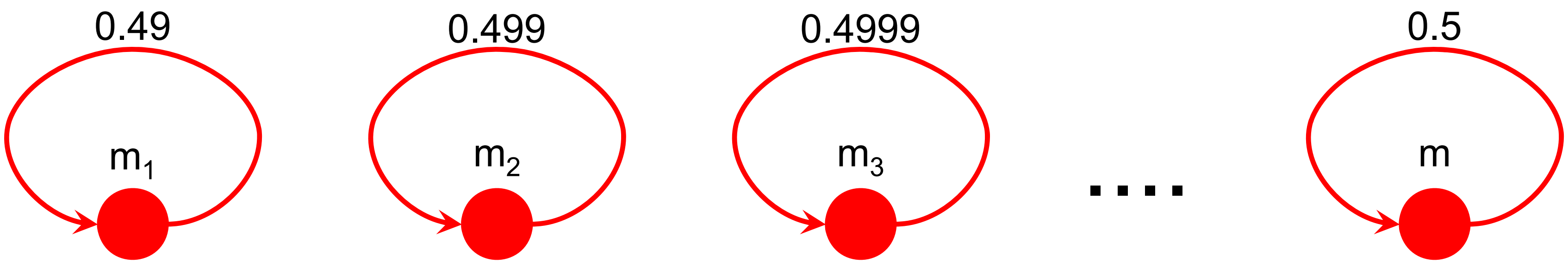


Mathematical perspective

A second attempt - Wrong quantifiers!

For $\theta : M \to \Delta(M,\Sigma)$, a "good" bisimilarity distance is a pseudometric $d: M \times M \to [0,1]$ such that for any sequence $(m_k)_k$ with $m_k \xrightarrow{d} m$,

- $\forall\, S \in \Sigma(\sim),\ \forall\, (S_k)_k \subseteq \Sigma(\sim)$ such that $S_k \xrightarrow{d^H} S$
    - $\theta(m_k)(S_k) \xrightarrow{\mathbb{R}} \theta(m)(S)$



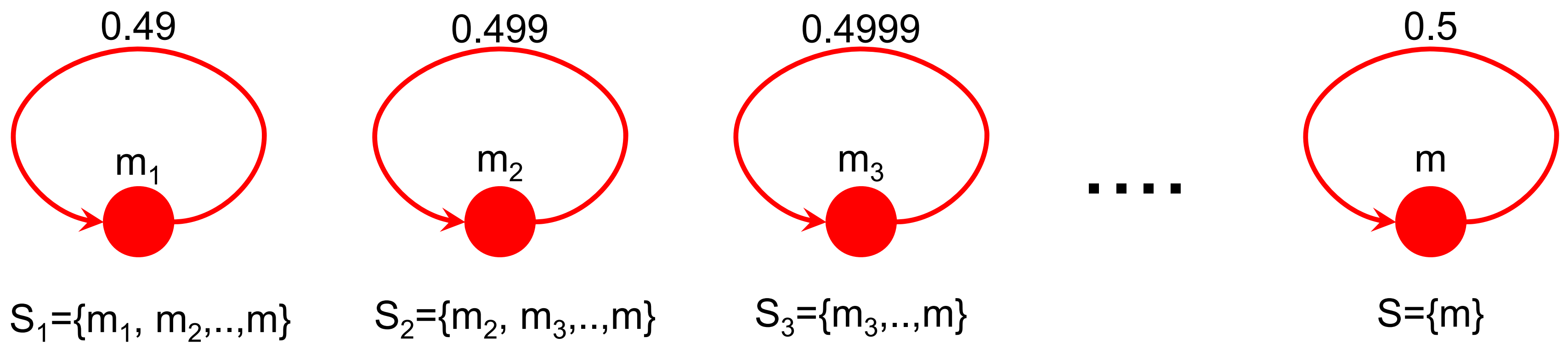
$S_1 = S_2 = \ldots = S = \{m\}$ $\quad\theta(m_k)(S)=0$ for any k and $\theta(m)(S)=0.5$

A third attempt

For $\theta : M \to \Delta(M,\Sigma)$, a dynamically-continuous bisimilarity distance is a pseudometric $d : M \times M \to [0,1]$ such that for any sequence $(m_k)_k$,

$m_k \xrightarrow{d} m$ implies

- $\forall S \in \Sigma(\sim), \exists (S_k)_k \subseteq \Sigma(\sim)$ such that
  - $S_k \xrightarrow{d^H} S$
  - $\theta(m_k)(S_k) \xrightarrow{\mathbb{R}} \theta(m)(S)$



Mathematical perspective

However, the concept of dynamic-continuity is not sufficient to solve our problem since the following distance

$$d(m,n) = \begin{cases} 0 & \text{if } m \sim n \\ 1 & \text{otherwise.} \end{cases}$$

is dynamic-continuous!

“Classic” Stone duality for MPs

A **Stone Markov process** is a Markov process $\theta : M \rightarrow \Delta(M,\Sigma)$, where

- Σ is the Borel algebra induced by a topology ↄ which is
  - Hausdorff
  - saturated in the sense of Model Theory (but not compact)
  - has a countable (designated) base of clopens closed under
    - set-theoretic Boolean operations
    - the operation $L_r c = \{m \mid \theta(m)(c) \leq r\}$

A **morphism of SMPs** is a morphism of MPs which, in addition,

- it is continuous
- preserves the designated bases forward and backward

An **Aumann Algebra** is a structure (A, $\rightarrow$, $\perp$, $\{L_r\}_{r\in\mathbb{Q}}$, $\sqsubseteq$) where,

- (A, $\rightarrow$, $\perp$, $\sqsubseteq$) is a Boolean algebra
- for $r\in\mathbb{Q}^+$, L_r: A→ A is an unary operator satisfying the axioms below

(AA1) $\top \sqsubseteq L_0 a$

(AA2) $\top \sqsubseteq L_r \top$

(AA3) $L_r a \sqsubseteq \neg L_s \neg a, \quad r+s>1$

(AA4) $L_r(a \wedge b) \wedge L_s(a \wedge \neg b) \sqsubseteq L_{r+s} a, \quad r+s \leq 1$

(AA5) $\neg L_r(a \wedge b) \wedge \neg L_s(a \wedge \neg b) \sqsubseteq \neg L_{r+s} a, \quad r+s \leq 1$

(AA6) $a \sqsubseteq b \Rightarrow L_r a \sqsubseteq L_r b$

(AA7) $\bigwedge_{r<s} L_{r_1 \cdots r_n r} a = L_{r_1 \cdots r_n s} a$

A **morphism of AAs** is a morphism of Boolean algebras that also preserves the L_r operators

“Classic” Stone duality for MPs

Given a countable AA

$$\mathscr{A} = (A, \rightarrow, \perp, \{L_r\}_{r\in\mathbb{Q}}, \sqsubseteq)$$

we can construct an SMP from the set of ultrafilters of $\mathscr{A}$

the topology is generated by the clopens of type

$$\{u - \text{utrafilter},\ a\in u\},\ \text{for all } a\in A.$$

We denote it by $\boldsymbol{\mathcal{M}}(\mathscr{A})$.

"Classic" Stone duality for MPs

Given an SMP with B its designated base,

$$\mathcal{M}=(M,\Sigma_B, \theta)$$

we can construct a countable AA from the set B of clopens,

with the operators L_r satisfying the following condition

$$c'=L_r c \text{ iff } c'=\{m \mid \theta(m)(c)\leq r\}.$$

We denote it by $\mathbf{A}(\mathcal{M})$.

Representation Theorem:

- Any countable Aumann Algebra $\mathcal{A}=(A, \rightarrow, \perp, \{L_r\}_{r\in\mathbb{Q}}, \sqsubseteq)$ is isomorphic to

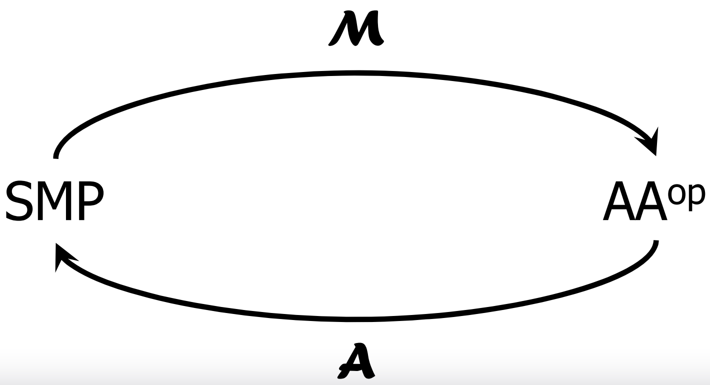
$\mathbf{A}(\mathbf{M}(\mathcal{A}))$ via the map $\beta: \mathcal{A} \to \mathbf{A}(\mathbf{M}(\mathcal{A}))$ defined by

$$\beta(a)=\{u \text{ – ultrafilter} \mid a \in u\}.$$

- Any Stone Markov process $\mathcal{M}=(M,\Sigma, \theta)$ is homeomorphic to

$\mathbf{M}(\mathbf{A}(\mathcal{M}))$ via the map $\alpha: \mathcal{M} \to \mathbf{M}(\mathbf{A}(\mathcal{M}))$ defined by

$$\alpha(m)=\{c \text{ – designated clopen} \mid m \in c\}.$$



[LICS 2013]

The lesson of the “classic” Stone duality for MPs

There exists a complex relationship between

$\mathcal{C}_B$ and $M^{\sim}$

This relation must be generalized to reflect the relationship between

$\mathcal{C}_B$ and $\mathcal{C}_d$

The lesson of the "classic" Stone duality for MPs

There exists a complex relationship between

$\mathcal{T}_B$ and $M^{\sim}$

This relation must be generalized to reflect the relationship between

$\mathcal{T}_B$ and $\mathcal{T}_d$

Theorem:

Given an SMP (M,B,θ) and a pseudometric $d:M \times M \to [0,1]$, the following statements are equivalent:

1. $\forall m, \inf_{c \in B,\, m \in c} \sup\{d(n,n') \mid n,n' \in c\}=0$
2. $\forall m, m' \inf_{c \in B,\, m,m' \in c} \sup\{d(n,n') \mid n,n' \in c\}=d(m,m')$
3. The topology $\mathcal{T}_B$ refines the topology $\mathcal{T}_d$
4. The pseudometric d is continuous in both arguments with respect to $\mathcal{T}_B$.

The lesson of the "classic" Stone duality for MPs

There exists a complex relationship between

τ_B and $M^{\sim}$

This relation must be generalized to reflect the relationship between

τ_B and τ_d

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2. $\forall m, m' \inf_{c \in B,\, m,m' \in c} \sup\{d(n,n') \mid n,n' \in c\} = d(m,m')$
3. The topology τ_B refines the topology τ_d
4. The pseudometric d is continuous in both arguments with respect to τ_B.

Moreover, the previous conditions enforce the concept of dynamic-continuity.

Metrized Stone duality for MPs

A **Metrized Markov process** is an SMP (M,B,θ) endowed with a pseudometric $d:M\times M\rightarrow[0,1]$ that satisfies, for arbitrary $m\in M$ the property

$$\text{(M)}\quad \inf_{m\in c\in B}\ \sup\{d(n,n') \mid n,n'\in c\}=0$$

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$$(M) \quad \inf_{m \in c \in B} \sup\{d(n,n') \mid n,n' \in c\}=0$$

Given two MMPs (M_i,B_i,θ_i,d_i) $i=1,2$, a **morphism of MMPs** is a non-expansive morphism $f: M_1 \rightarrow M_2$ of SMPs, i.e. such that

$$d_1(m,n) \geq d_2(f(m),f(n))$$

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Given two MMPs (M_i,B_i,θ_i,d_i) $i=1,2$, an **isometry** between them is a mapping $f:M_1\rightarrow M_2$ such that

$$d_1(m,n)=d_2(f(m),f(n))$$

Metrized Stone duality for MPs

A **Metrized Aumann Algebra** is an AA $(A, \rightarrow, \perp, \{L_r\}_{r\in\mathbb{Q}}, \sqsubseteq)$ endowed with a concept of diameter of its elements

$$| \ | : A \rightarrow [0,1]$$

that satisfies the following axioms

- $|\perp|=0$
- $a \sqsubseteq b$ implies $|a| \leq |b|$
- if $a \wedge b \neq \perp$, then $|a \vee b| \leq |a|+|b|$
- for any ultrafilter u, $\inf\{|a| , a \in u\}=0$

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Given two MAAs $(A_i, | \ |_i)$, i=1,2 , a **morphism of MAAs** is an expansive morphism $f: A_1 \rightarrow A_2$ of AAs, i.e., such that

$$|a|_1 \leq |f(a)|_2$$

Metrized Stone duality for MPs

A **Metrized Aumann Algebra** is an AA $(A, \to, \bot, \{L_r\}_{r\in\mathbb{Q}}, \sqsubseteq)$ endowed with a concept of diameter of its elements

$$|\ \ |: A \to [0,1]$$

that satisfies the following axioms

- $|\bot|=0$
- $a \sqsubseteq b$ implies $|a| \leq |b|$
- if $a \wedge b \neq \bot$, then $|a \vee b| \leq |a|+|b|$
- for any ultrafilter u, $\inf\{|a| , a\in u\}=0$

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$$|a|_1 = |f(a)|_2$$

Metrized Stone duality for MPs

Given a countable MAA $\mathscr{A}$= (A,| |), we can construct an MMP by endowing $\mathcal{M}(\mathscr{A})$ with the pseudometric $\boldsymbol{\delta}_{|\,|}$ defined over the set of ultrafilters of A as follows

$$\boldsymbol{\delta}_{|\,|}(u,v)=\inf\{|a|,\ a\in u\cap v\}.$$

Metrized Stone duality for MPs

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$$\boldsymbol{\delta}_{|\ |}(u,v)=\inf\{|a|,\ a\in u\cap v\}.$$

Given an MMP $\mathscr{M}=(M,\Sigma_B,\theta,d)$ with B its base, we can construct a countable MAA by endowing $\boldsymbol{A}(\mathscr{M})$ with a diameter $|\ |_d$ defined on B as follows

$$|\ |_d(c)=\inf\{d(m,n),\ m,n\in c\}.$$

Metrized Stone duality for MPs

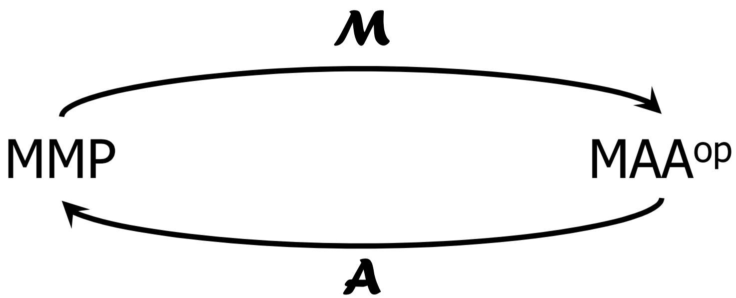
Extended Representation Theorem:

1. Any countable Metrized Aumann Algebra $\mathscr{A}=(A, \to, \bot, \{L_r\}_{r\in\mathbb{Q}}, \sqsubseteq, |\ |)$ is isomorphic to $\boldsymbol{A}(\boldsymbol{M}(\mathscr{A}))$ via the map β: $\mathscr{A} \to \boldsymbol{A}(\boldsymbol{M}(\mathscr{A}))$ defined by

$$\beta(a)=\{u - \text{ultrafilter} \mid a\in u\}.$$

Moreover, β is an isometry of MAAs, i.e.,

$$|a|=|\beta(a)|_{\boldsymbol{\delta}_{||}}$$



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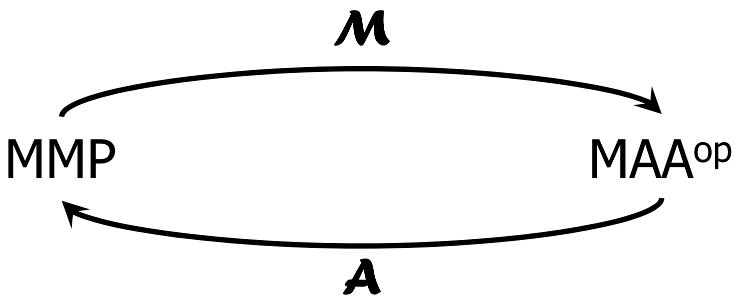
Extended Representation Theorem:

2. Any Stone Markov process $\mathscr{M}=(M,\Sigma,\theta)$ is homeomorphic to $\boldsymbol{\mathcal{M}}(\boldsymbol{\mathcal{A}}(\mathscr{M}))$ via the map $\alpha: \mathscr{M} \to \boldsymbol{\mathcal{M}}(\boldsymbol{\mathcal{A}}(\mathscr{M}))$ defined by

$$\alpha(m)=\{c \text{ – designated clopen} \mid m \in c\}.$$

Moreover, α is an isometry of MMPs, i.e.,

$$d(m,n)=\delta_{\|\,\|_d}(\alpha(m),\alpha(n))$$



Conclusions

- We provide a characterization of the behavioural distances that induce well-behaved topologies.

- The “classic” Stone duality for MPs are not only clarifying the relation between MPs, Markovian logics and bisimilarity, but they also provide the right framework that allows us to extend the bisimilarity-based semantics to a distance-based semantics.

- The relation between bisimilarity classes and the support topology of an SMP can be generalized to understand the relation between the same topology and the open-ball topology induced by a behavioral distance.

- The metric duality underlines the importance of a concept of “diameter” for the elements of the Boolean algebra.