Markovian Logics: Completeness and Dualities

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Markov Processes

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Given an analytic space (M, Σ), a *Markov process* is a measurable mapping

$$\theta : M \longrightarrow \Pi(M, \Sigma) \quad \text{– probabilistic case}$$
$$\theta : M \longrightarrow \Pi^*(M, \Sigma) \quad \text{– subprobabilistic case}$$
$$\theta : M \longrightarrow \Delta(M, \Sigma) \quad \text{– stochastic case}$$

- $\Pi(M, \Sigma)$ – probabilistic distributions on (M, Σ)
- $\Pi^*(M, \Sigma)$ – subprobabilistic distributions on (M, Σ)
- $\Delta(M, \Sigma)$ – general distributions on (M, Σ)

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The measurable space of distributions is generated by sets

$$\{\mu \in \Delta(M, \Sigma) \mid \mu(A) \leq r\}$$

defined for arbitrary $A \in \Sigma$ and $r \in \mathbb{Q}$.

Markovian Logics

Syntax:

$$\mathcal{L}(\Pi), \mathcal{L}(\Pi^*) : \quad \phi ::= p \in \mathcal{P} \mid \bot \mid \phi \to \phi \mid L_r\phi, \qquad r \in \mathbb{Q} \cap [0,1]$$

$$\mathcal{L}(\Delta) : \quad \phi ::= p \in \mathcal{P} \mid \bot \mid \phi \to \phi \mid L_r\phi, \qquad r \in \mathbb{Q}^+$$

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Semantics:

$\mathcal{M} = (M, \Sigma, \theta)$, $m \in M$ and $i : M \to 2^{\mathcal{P}}$,

The satisfaction relation:

- $\mathcal{M}, m, i \models p$ if $p \in i(m)$,
- $\mathcal{M}, m, i \models \bot$ never,
- $\mathcal{M}, m, i \models \phi \to \psi$ if $\mathcal{M}, m, i \models \psi$ whenever $\mathcal{M}, m, i \models \phi$,
- $\mathcal{M}, m, i \models L_r\phi$ if $\theta(m)([\![\phi]\!]) \geq r$,
  where $[\![\phi]\!] = \{m \in M \mid \mathcal{M}, m, i \models \phi\}$.

Axioms - probabilistic case

The axioms of $\mathcal{L}(\Pi)$

(A1): $\vdash L_0\phi$

(A2): $\vdash L_r T$

(A3): $\vdash L_r\phi \longrightarrow \neg L_s\neg\phi, \quad r+s>1$

(A4): $\vdash L_r(\phi \wedge \psi) \wedge L_s(\phi \wedge \neg\psi) \longrightarrow L_{r+s}\phi, \quad r+s \leq 1$

(A5): $\vdash \neg L_r(\phi \wedge \psi) \wedge \neg L_s(\phi \wedge \neg\psi) \longrightarrow \neg L_{r+s}\phi, \quad r+s \leq 1$

(R1): $\dfrac{\vdash \phi \longrightarrow \psi}{\vdash L_r\phi \longrightarrow L_r\psi}$

(R2): $\{L_r\psi \mid r < s\} \vdash L_s\psi$

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(R2): $\{L_r\psi \mid r<s\}\vdash L_s\psi$

Weak Completeness

$\mathcal{L}(\Pi)$ is sound and weak-complete for the probabilistic Markov processes

$$\models \phi \text{ iff } \vdash \phi.$$

Axioms - subprobabilistic case

The axioms of $\mathcal{L}(\Pi^*)$

(A1): $\vdash L_0\phi$

(A2'): $\vdash L_r\bot \to \bot$

(A3): $\vdash L_r\phi \to \neg L_s\neg\phi, \quad r+s>1$

(A4): $\vdash L_r(\phi\wedge\psi)\wedge L_s(\phi\wedge\neg\psi) \to L_{r+s}\phi, \quad r+s\leq 1$

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$\mathcal{L}(\Pi^*)$ is sound and weak-complete for the subprobabilistic Markov processes

$$\models \phi \text{ iff } \vdash \phi.$$

Axioms - stochastic case

The axioms of $\mathcal{L}(\Delta)$

(A1): $\vdash L_0\phi$

(A2'): $\vdash L_r\bot \rightarrow \bot$

(A4'): $\vdash L_r(\phi \wedge \psi) \wedge L_s(\phi \wedge \neg\psi) \rightarrow L_{r+s}\phi$

(A5'): $\vdash \neg L_r(\phi \wedge \psi) \wedge \neg L_s(\phi \wedge \neg\psi) \rightarrow \neg L_{r+s}\phi$

(R1): $\dfrac{\vdash \phi \rightarrow \psi}{\vdash L_r\phi \rightarrow L_r\psi}$

(R2): $\{L_r\psi \mid r < s\} \vdash L_s\psi$

(R3): $\{L_r\psi \mid r \in \mathbb{Q}^+\} \vdash \bot$

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(A4'): $\vdash L_r(\phi \wedge \psi) \wedge L_s(\phi \wedge \neg\psi) \rightarrow L_{r+s}\phi$
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(R1): $\dfrac{\vdash \phi \rightarrow \psi}{\vdash L_r\phi \rightarrow L_r\psi}$
(R2): $\{L_r\psi \mid r < s\} \vdash L_s\psi$
(R3): $\{L_r\psi \mid r \in \mathbb{Q}^+\} \vdash \bot$

Weak Completeness

$\mathcal{L}(\Delta)$ is sound and weak-complete for the stochastic Markov processes

$$\models \phi \text{ iff } \vdash \phi.$$

Compactness of Markovian Logics

$$(R2): \quad \{L_r\psi \mid r < s\} \vdash L_s\psi$$

$\mathcal{L}(\Pi)$, $\mathcal{L}(\Pi^*)$ and $\mathcal{L}(\Delta)$ are not compact:

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for a consistent formula ϕ, the set

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is inconsistent (due to R2), but all its finite subsets are consistent.

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is inconsistent (due to R2), but all its finite subsets are consistent.

Consequently, the logics are not necessarily strongly complete,
i.e., we might need extra axioms to prove that

$$\Phi \models \phi \text{ iff } \Phi \vdash \phi,$$

for arbitrary $\Phi \subseteq \mathcal{L}$ and $\phi \in \mathcal{L}$.

Strong Completeness

[Goldblatt, J. Logic Comput. 2010]

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- the semantic consequence relation over T-coalgebras is equal to the least deducibility relation that satisfies Lindenbaum's lemma.

Strong Completeness

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- the logic of T-coalgebras – T measurable polynomial functor on the category of measurable spaces;
- the semantic consequence relation over T-coalgebras is equal to the least deducibility relation that satisfies Lindenbaum's lemma.
- Moreover, strong completeness – requires a strengthened version of (R1) - the countable additivity rule (CAR):

For Φ – closed under conjunction,

$$\text{(CAR):} \quad \frac{\Phi \vdash \phi}{L_r\Phi \vdash L_r\phi}$$

where $L_r\Phi = \{L_r\psi \mid \psi \in \Phi\}$.

Strong Completeness Proofs

[Zhou, J. Logic Lang. and Comput. 2010]

- Goldblatt's (CAR) rule and Lindenbaum's lemma => strong completeness of probabilistic logic for Harsanyi type spaces;
- proves that without Goldblatt's rule the logic is not complete.

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[Cardelli, Mardare, Larsen, ICALP2011, CSL2011, LMCS 2012]

- similar systems => the strong completeness for various logics on general Markov processes.

Strong Completeness

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Observe that (CAR) has uncountably many instances.

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$$\text{(CAR):} \quad \frac{\Phi \vdash \phi}{L_r\Phi \vdash L_r\phi}$$

Observe that (CAR) has uncountably many instances.

Consequently, one cannot simply use the Zorn's lemma to prove that any consistent set of formulas can be extended to a maximally consistent set.

Proving Lindenbaum's property is highly non-trivial!

The Rasiowa-Sikorski Lemma

Let $\mathcal{B}$ be a Boolean algebra and $T \subset \mathcal{B}$ be a set with $\bigwedge T$ defined. An ultrafilter U is said to *respect* T if

$$T \subseteq U \Rightarrow \bigwedge T \in U.$$

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Let $\mathcal{T}$ be a countable family of subsets of $\mathcal{B}$ each member of which has a meet in $\mathcal{B}$ and let $x \neq 0$. There exists an ultrafilter which respects each member of $\mathcal{T}$ and which contains x.

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Corollary

Given a Boolean logic with a countable axiomatization. Any consistent set of formulas can be extended to a maximally consistent set that respects all the instances of the axioms and rules.

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(R2'): $\{L_{r_1\cdots r_n r}\psi \mid r<s\} \vdash L_{r_1\cdots r_n s}\psi$

where $L_{r_1\cdots r_n r}\psi = L_{r_1}L_{r_2}..L_{r_n}L_r\psi$.

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where $L_{r_1\cdots r_n r}\psi = L_{r_1}L_{r_2}..L_{r_n}L_r\psi$.

We prove the strong completeness for $\mathcal{L}(\Pi)$

Aumann Algebra - the probabilistic case

[Kozen, Larsen, Mardare, Panangaden, LICS2013]

A (probabilistic) Aumann algebra is a structure

$$\mathcal{A} = (A, \to, \bot, \{L_r\}_{r\in\mathbb{Q}\cap[0,1]}, \sqsubseteq)$$

- $(A, \to, \bot, \sqsubseteq)$ is a Boolean algebra;
- $L_r : A \to A$ is an unary operator, for $r \in \mathbb{Q} \cap [0, 1]$.

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(AA1) $\top \sqsubseteq L_0 a$

(AA2) $\top \sqsubseteq L_r \top$

(AA3) $L_r a \sqsubseteq \neg L_s \neg a, \quad r + s > 1$

(AA4) $L_r(a \wedge b) \wedge L_s(a \wedge \neg b) \sqsubseteq L_{r+s} a, \quad r + s \leq 1$

(AA5) $\neg L_r(a \wedge b) \wedge \neg L_s(a \wedge \neg b) \sqsubseteq \neg L_{r+s} a, \quad r + s \leq 1$

(AA6) $a \sqsubseteq b \Rightarrow L_r a \sqsubseteq L_r b$

(AA7) $\bigwedge_{r<s} L_{r_1\cdots r_n r} a = L_{r_1\cdots r_n s} a$

Markovian logic yields an Aumann algebra

Let $[\phi]$ denote the equivalence class of ϕ modulo $\equiv$, and let $\mathcal{L}(\Pi)/{\equiv} = \{[\phi] \mid \phi \in \mathcal{L}\}$.

Theorem

The structure

$$(\mathcal{L}(\Pi)/{\equiv}, \rightarrow, [\bot], \{L_r\}_{r \in \mathbb{Q}_0}, \leq)$$

is a countable probabilistic Aumann algebra,
where $[\phi] \leq [\psi]$ iff $\vdash \phi \rightarrow \psi$.

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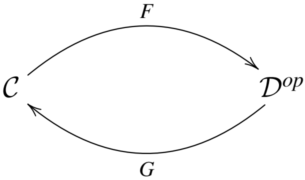
filters of AA ==> consistent sets of $\mathcal{L}(\Pi)$

ultrafilters of AA ==> maximal consistent sets of $\mathcal{L}(\Pi)$

Recap of Stone Duality



Stone Duality

We have a (contravariant) adjunction between categories $\mathcal{C}$ and $\mathcal{D}$, which is an *equivalence* of categories.

Examples: Finite sets and finite Boolean algebras, Boolean algebras and Stone spaces, Finite-dimensional vector spaces and itself, commutative unital C^*-algebras and compact Hausdorff spaces, ...

Reasoning on ultrafilters

Fix an arbitrary countable Aumann algebra

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- Let $\mathcal{U}^*$ be the set of all Boolean ultrafilters of $\mathcal{A}$.
- The Stone duality construction for Boolean algebras with operators [Jonsson-Tarski, Am. J. of Math. 1951]: a Boolean algebra of sets isomorphic to $\mathcal{A}$ with elements

$$(\!|a|\!)^* = \{u \in \mathcal{U}^* \mid a \in u\}, \; a \in \mathcal{A}$$
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- The sets $(\!|a|\!)^*$ are the clopen sets that generate a Stone topology τ^* on $\mathcal{U}^*$ <== compact, zero-dimensional, Hausdorff space.

Ultrafilters: Good and Bad

Recall the axiom

$$\text{(AA7)} \quad \bigwedge_{r<s} L_{r_1\cdots r_n r}a = L_{r_1\cdots r_n s}a$$

It is the only infinitary axiom-schema:

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It is the only infinitary axiom-schema:

- some Boolean ultrafilers in $\mathcal{U}^*$ respect all the instances of this axiom – the good ultrafilters **<== Rasiowa- Sikorski Lemma**
- some ultrafilters in $\mathcal{U}^*$ violates one or more instances of (AA7) – the bad ultrafilters.

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- some Boolean ultrafilers in $\mathcal{U}^*$ respect all the instances of this axiom – the good ultrafilters **<== Rasiowa- Sikorski Lemma**
- some ultrafilters in $\mathcal{U}^*$ violates one or more instances of (AA7) – the bad ultrafilters.

Let $\mathcal{U}$ be the set of good ultrafilters of $\mathcal{A}$ and

$$(\!|a|\!) = \{u \in \mathcal{U} \mid a \in u\},\ a \in \mathcal{A}$$
$$(\!|\mathcal{A}|\!) = \{(\!|a|\!) \mid a \in A\}.$$

Then $\mathcal{U}^* \setminus \mathcal{U}$ is the set of bad ultrafilters.

The space of good ultrafilters

We proved the following results:

1. The set $\mathcal{U}$ of good ultrafilters is dense in the set $\mathcal{U}^*$ of all ultrafilters; the set $\mathcal{U}^* \setminus \mathcal{U}$ of bad ultrafilters is meager in the Stone topology.

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2. Since $\mathcal{U}^*$ is a Stone space, the subspace $\mathcal{U}$ of good ultrafilters with the subspace topology is a zero-dimensional Hausdorff space.

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2. Since $\mathcal{U}^*$ is a Stone space, the subspace $\mathcal{U}$ of good ultrafilters with the subspace topology is a zero-dimensional Hausdorff space.
3. However, $\mathcal{U}$ is not compact any more; but it is saturated in the sense of Model Theory.

The space of the cannonic model

- Consider the subspace topology of the good ulrafilters

$$(\mathcal{U}, \tau) \subset (\mathcal{U}^*, \tau^*).$$

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- The Borel algebra induced by τ coincides with the σ-algebra generated by the field of sets $(\!|\mathcal{A}|\!)$.
- Hence, $(\mathcal{U}, (\!|\mathcal{A}|\!)^\sigma)$ is a measurable space.
- Moreover, $(\mathcal{U}, (\!|\mathcal{A}|\!)^\sigma)$ is an analytic space, since τ is Hausdorff, saturated and zero-dimensional.

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- Hence, $(\mathcal{U}, (\!|A|\!)^\sigma)$ is a measurable space.
- Moreover, $(\mathcal{U}, (\!|A|\!)^\sigma)$ is an analytic space, since τ is Hausdorff, saturated and zero-dimensional.

$\mathcal{U}$ is used as the state space for (the cannonic) Markov process.

The transition of the cannonic model

1. For any good ultrafilter $u \in \mathcal{U}$ and any $a \in A$,

$$\sup\{r \mid L_r a \in u\} = \inf\{r \mid \neg L_r a \in u\}.$$

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3. The set function $\theta(u)$ is finitely additive and continuous from above at $\emptyset$ on the field $\llparenthesis \mathcal{A} \rrparenthesis$ of sets **<== Rasiowa-Sikorski Lemma.**

4. One can use standard measure extension theorems to define θ as a measure on the measurable space $(\mathcal{U}, \llparenthesis \mathcal{A} \rrparenthesis^{\sigma})$ of good ultrafilters.

The cannonic model

The Markov process of good ultrafilters

If $\mathcal{A}$ is a countable Aumann algebra, then we can construct a countably-generated Markov process, $\mathbb{M}(\mathcal{A}) = (\mathcal{U}, (\!|\mathcal{A}|\!)^{\sigma}, \theta)$, on the space $\mathcal{U}$ of good ultrafilters.
Moreover, $(\!|\mathcal{A}|\!)$ is the base of a topology that is

- zero-dimensional,
- Hausdorff,
- saturated.

The cannonic model

The Markov process of good ultrafilters

If $\mathcal{A}$ is a countable Aumann algebra, then we can construct a countably-generated Markov process, $\mathbb{M}(\mathcal{A}) = (\mathcal{U}, (\!|\mathcal{A}|\!)^{\sigma}, \theta)$, on the space $\mathcal{U}$ of good ultrafilters.
Moreover, $(\!|\mathcal{A}|\!)$ is the base of a topology that is

- zero-dimensional,
- Hausdorff,
- saturated.

Truth Lemma

Let $\Phi \subseteq \mathcal{L}$ be an arbitrary theory and $u \subseteq \mathcal{L}$ an arbitrary maximal consistent set of $\mathcal{L}$ – observe that $u \in \mathcal{U}(\mathcal{L})$.
Then,

$$\Phi \subseteq u \quad \text{iff} \quad \mathbb{M}(\mathcal{L}), u \models \Phi.$$

Strong Completeness for Markovian logics

Strong Completeness

Let $\Phi \subseteq \mathcal{L}$ be an arbitrary theory and $\phi \in \mathcal{L}$. Then,

$$\Phi \models \phi \quad \text{iff} \quad \Phi \vdash \phi.$$

The result applies to $\mathcal{L} \in \{\mathcal{L}(\Pi), \mathcal{L}(\Pi^*), \mathcal{L}(\Delta)\}$ with the corresponding semantics and axiomatization.

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- $\mathcal{D}^\sigma$ - the Borel algebra of the topology induced by $\mathcal{D}$.
- An SMP $(M, \mathcal{D}^\sigma, \theta)$ is an MP defined on such a structure.
- Morphisms of SMPs are *continuous* function $f : \mathcal{M} \to \mathcal{N}$ s.t.
  1. for $B \in \mathcal{D}^\sigma_{\mathcal{N}}$, $\theta_{\mathcal{M}}(m)(f^{-1}(B)) = \theta_{\mathcal{N}}(f(m))(B)$;
  2. for $D \in \mathcal{D}^\sigma_{\mathcal{N}}$, $f^{-1}(D) \in \mathcal{D}^\sigma_{\mathcal{M}}$.

The Aumann algebra of clopens

Let $\mathcal{M} = (M, \mathcal{B}^{\sigma}, \theta)$ be a Stone Markov process.

The Aumann algebra of clopens

The structure $\mathcal{B}$ with the set-theoretic Boolean operations and the operations L_r for $r \in \mathbb{Q} \cap [0, 1]$, is a countable Aumann algebra.

We denote this algebra by $\mathbb{A}(\mathcal{M})$.

The duality

We defined two contravariant functors:

$\mathbb{A}(\cdot) : \mathbf{SMP} \longrightarrow \mathbf{AA}^{\text{op}}$

On arrows $f : \mathcal{M} \longrightarrow \mathcal{N}$ we define $\mathbb{A}(f) = f^{-1} : \mathbb{A}(\mathcal{N}) \longrightarrow \mathbb{A}(\mathcal{M})$.

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On morphisms $h : \mathcal{A} \to \mathcal{B}$, $\mathbb{M}(h) = h^{-1} : \mathbb{M}(\mathcal{B}) \to \mathbb{M}(\mathcal{A})$, explicitly

$$\mathbb{M}(h)(u) = h^{-1}(u) = \{A \in \mathcal{A}_{\mathcal{N}} \mid h(A) \in u\}.$$

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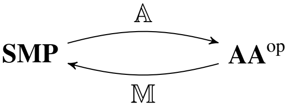
The functors $\mathbb{M}$ and $\mathbb{A}$ define a dual equivalence of categories.



Representation

The representation theorem

1. Any countable Aumann algebra $\mathcal{A} = (A, \longrightarrow, \perp, \{L_r\}_{r\in\mathbb{Q}^+}, \sqsubseteq)$ is isomorphic to $\mathbb{A}(\mathbb{M}(\mathcal{A}))$ via the map $\beta : \mathcal{A} \to \mathbb{A}(\mathbb{M}(\mathcal{A}))$ defined by

$$\beta(a) = \{u \in \mathrm{supp}(\mathbb{M}(\mathcal{A})) \mid a \in u\} = (\!|a|\!).$$

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$$\beta(a) = \{u \in \mathrm{supp}(\mathbb{M}(\mathcal{A})) \mid a \in u\} = (\!|a|\!).$$

2. Any saturated Markov process $\mathcal{M} = (M, \mathcal{A}, \theta)$ is homeomorphic to $\mathbb{M}(\mathbb{A}(\mathcal{M}))$ via the map $\alpha : \mathcal{M} \to \mathbb{M}(\mathbb{A}(\mathcal{M}))$ defined by

$$\alpha(m) = \{A \in \mathcal{A} \mid m \in A\}.$$

What did we get?

- The Stone duality for MPs explains the intrinsic relationship between the category of **(Stone) Markov processes** and the category of **(countable) Aumann algebras.**

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- The Stone duality for MPs explains the intrinsic relationship between the category of **(Stone) Markov processes** and the category of **(countable) Aumann algebras**.
- The logical characterization of bisimilarity reflects the fact that the separability relation induced by the support topology of an SMP coincides to the bisimilarity relation.
- Because this topology has a base formed from positive formulas, we can characterize the bisimilarity considering only the negation-free formulas.
- Similarly, we could characterize bisimilarity using any other base.

What's next?

- Bisimilarity is too restrictive ==> **bisimilarity distances** that measure how similar two non-bisimilar MPs are.

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- We have discovered that **a metric version of the Stone duality** actually exists when we move from the bisimulation-based semantics for MPs to the distance-based one.

More details in our presentation today from 17:20:

A Metric Analog of Stone Duality for Markov Processes
Kozen, Mardare, Panangaden

Saturation

Notice in the previous definition that

the space is not required to be compact.

- We aim to compensate this by introducing a concept of "saturation" similar to the one used in Model Theory;
- Intuitively, one adds points to the structure without changing the represented algebra. An MP is *saturated* if it is maximal with respect to this operation.

Our contribution to the Strong Completeness Proof

Goldblatt's rule can be replaced with an infinitary rule having a countable set of instances.

Our infinitary rule allows us to apply the Rasiowa-Sikorski Lemma to prove Lindenbaum's Lemma and strong completeness.

This result can be generalized to a Stone duality theorem

[Kozen, Larsen, Mardare, Panangaden, LICS2013.]

The proof technique can be applied to logics for measurable polynomial functors on the category of measurable spaces as well as in other contexts of non-compact modal logics with normal modal operators.

Our results rely on some subtle topological issues that are the cornerstone of this work.