Labelled Markov Processes

Logical characterization of bisimulation for LMPs

Josée Desharnais[1]
Abbas Edalat, Prakash Panangaden
Vineet Gupta, Radha Jagadeesan

[1]Laval University Québec, Canada

MFPS, Cornell University, June 2014

Outline

What are Labelled Markov Processes?

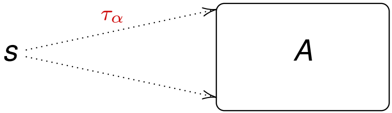
LMPs are

- probabilistic versions of labelled transition systems.
- probabilistic data is internal
- we observe the interactions - not the internal states.
- **the state space may be a continuum.**

Formal Definition of LMPs

An LMP is a tuple $(S, \Sigma, L, \forall \alpha \in L.\ \tau_\alpha)$

- (S, Σ) is an **analytic space**
- L is a countable set of labels
- $\tau_\alpha : S \times \Sigma \longrightarrow [0, 1]$ is a stochastic kernel,



$\tau_\alpha(s, S) \leq 1$

What is measure theory?

We want to assign a “size” to sets so that we can use it for quantitative purposes, like integration or probability.

Examples (of known measures)

- the size of an interval $[\pi/2, \pi]$, the area of a figure
- the probability of events when rolling a regular die

What is measure theory?

We want to assign a “size” to sets so that we can use it for quantitative purposes, like integration or probability.

Examples (of known measures)

- the size of an interval $[\pi/2, \pi]$, the area of a figure
- the probability of events when rolling a regular die

- Counting points is useless for the continuum.
- What is the “length” of the rational numbers in $[0, 1]$?

What is measure theory?

We want to assign a "size" to sets so that we can use it for quantitative purposes, like integration or probability.

Examples (of known measures)

- the size of an interval $[\pi/2, \pi]$, the area of a figure
- the probability of events when rolling a regular die

- Counting points is useless for the continuum.
- What is the "length" of the rational numbers in $[0, 1]$?
- We want to assign sizes to these and (all?) other sets.

What are measurable sets anyway?

- Alas! Not all sets can be given a sensible notion of size that generalizes the notion of length of an interval.
- We take a family of sets satisfying “reasonable” axioms and deem them to be “measurable.”

Measurable space (X, Σ)

A measurable space (X, Σ) is a set X together with a family Σ of subsets of X, called a σ-algebra or σ-**field**

Definition (σ-algebra)

$\Sigma \subseteq \mathcal{P}(X)$ is a σ-algebra if

1. $\emptyset \in \Sigma$,
2. $A \in \Sigma$ implies that $A^c \in \Sigma$, and

Measurable space (X, Σ)

A measurable space (X, Σ) is a set X together with a family Σ of subsets of X, called a σ-algebra or **σ-field**

Definition (σ-algebra)

$\Sigma \subseteq \mathcal{P}(X)$ is a σ-algebra if

1. $\emptyset \in \Sigma$,
2. $A \in \Sigma$ implies that $A^c \in \Sigma$, and
3. if $\{A_i \in \Sigma | i \in I\}$ is a countable family then $\cup_{i \in I} A_i \in \Sigma$.

Measurable space (X, Σ)

A measurable space (X, Σ) is a set X together with a family Σ of subsets of X, called a σ-algebra or σ**-field**

Definition (σ-algebra)

$\Sigma \subseteq \mathcal{P}(X)$ is a σ-algebra if

1. $\emptyset \in \Sigma$,
2. $A \in \Sigma$ implies that $A^c \in \Sigma$, and
3. if $\{A_i \in \Sigma | i \in I\}$ is a countable family then $\cup_{i \in I} A_i \in \Sigma$.

If we require only finite union rather than countable union we get a **field** or **algebra**.

The σ-algebras generated by a family of sets

- Any intersection of σ-algebras is a σ-algebra.
- Thus given any family of sets $\mathcal{B}$ there is a least σ-algebra containing $\mathcal{B}$: the σ-algebra generated by $\mathcal{B}$, noted $\sigma(\mathcal{B})$.

  σ(intervals in $\mathbf{R}$) is called the Borel σ-algebra.

The σ-algebras generated by a family of sets

- Any intersection of σ-algebras is a σ-algebra.
- Thus given any family of sets $\mathcal{B}$ there is a least σ-algebra containing $\mathcal{B}$: the σ-algebra generated by $\mathcal{B}$, noted $\sigma(\mathcal{B})$.

  σ(intervals in $\mathbf{R}$) is called the Borel σ-algebra.

Measure on a measurable space (S, Σ)

Definition

A measure (probability measure) on (S, Σ) is a set function

$$\mu : \Sigma \longrightarrow [0, \infty] \quad ([0, 1]),$$

s.t. if $\{A_i\}_{i \in I}$ is a countable family of pairwise disjoint sets then

$$\mu\left(\bigcup_{i \in I} A_i\right) = \sum_{i \in I} \mu\left(A_i\right).$$

In particular if I is empty we have $\mu(\emptyset) = 0$.
and $\mu(A^c) = \mu(S) - \mu(A)$

Measure on a measurable space (S, Σ)

Definition

A measure (probability measure) on (S, Σ) is a set function

$$\mu : \Sigma \longrightarrow [0, \infty] \quad ([0, 1]),$$

s.t. if $\{A_i\}_{i \in I}$ is a countable family of pairwise disjoint sets then

$$\mu \left(\bigcup_{i \in I} A_i \right) = \sum_{i \in I} \mu \left(A_i \right).$$

In particular if I is empty we have $\mu(\emptyset) = 0$.
and $\mu(A^c) = \mu(S) - \mu(A)$

The structure (S, Σ, μ) is called a **measure space**.

Measurable sets are complicated beasts, we often want to work with families of simpler sets that generate the σ-algebra.

Corollary (to Dynkin's λ-π theorem)

Two measures on (S, Σ) that agree on a π-system $\mathcal{F} \subseteq \Sigma$ (closed under $\cap$) agree on $\sigma(\mathcal{F})$.

Measurable sets are complicated beasts, we often want to work with families of simpler sets that generate the σ-algebra.

Corollary (to Dynkin's λ-π theorem)

Two measures on (S, Σ) that agree on a π-system $\mathcal{F} \subseteq \Sigma$ (closed under $\cap$) agree on $\sigma(\mathcal{F})$.

Let

$$cl_{\Sigma}(\mathcal{F}) := \{A \in \Sigma \mid \text{ if } s \in A \text{ and } s \equiv_{\mathcal{F}} s' \text{ then } s' \in A\} \supseteq \sigma(\mathcal{F})$$

Measurable sets are complicated beasts, we often want to work with families of simpler sets that generate the σ-algebra.

Corollary (to Dynkin's λ-π theorem)

Two measures on (S, Σ) that agree on a π-system $\mathcal{F} \subseteq \Sigma$ (closed under $\cap$) agree on $\sigma(\mathcal{F})$.

Let

$$\begin{aligned} cl_\Sigma(\mathcal{F}) &:= \{A \in \Sigma \mid \text{ if } s \in A \text{ and } s \equiv_{\mathcal{F}} s' \text{ then } s' \in A\} \\ &\supseteq \sigma(\mathcal{F}) \end{aligned}$$

Theorem (1) (DP: JLAP03)

Let (S, Σ) be an analytic space, and $\mathcal{F}$ with $S \in \mathcal{F}$, countable and closed under intersection.
If two measures on (S, Σ) agree on $\mathcal{F} \subseteq \Sigma$, then they agree on $cl_\Sigma(\mathcal{F})$.

Functions

What are the “right” functions between measurable spaces?

$$f : (X, \Sigma_X) \to (Y, \Sigma_Y)$$

- inverse image preserve $\emptyset$, complement and unions

Functions

What are the “right” functions between measurable spaces?

$$f : (X, \Sigma_X) \longrightarrow (Y, \Sigma_Y)$$

- inverse image preserve $\emptyset$, complement and unions
- thus σ-algebras behave well under inverse image.

  $\{f^{-1}(A) | A \in \Sigma_Y\}$ form a σ-algebra on X.

Functions

What are the "right" functions between measurable spaces?

$$f : (X, \Sigma_X) \to (Y, \Sigma_Y)$$

- inverse image preserve $\emptyset$, complement and unions
- thus σ-algebras behave well under inverse image.

  $\{f^{-1}(A) | A \in \Sigma_Y\}$ form a σ-algebra on X.

Definition

A function f from a measurable space (X, Σ_X) to a measurable space (Y, Σ_Y) is said to be measurable if

$$f^{-1}(A) \in \Sigma_X \text{ whenever } A \in \Sigma_Y.$$

An example on $(X, \mathcal{P}(X))$

Fix a set X and a point x of X. We define a measure, in fact a probability measure, on the σ-algebra of all subsets of X as follows. We use the slightly peculiar notation $\delta(x, A)$ to emphasize that x is a parameter in the definition.

$$\delta(x, A) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

This measure is called the Dirac delta measure. Note that we can fix the set A and view this as the definition of a (measurable) function on X. What we get is the characteristic function of the set A, χ_A.

Lebesgue measure on $\mathbf{R}$

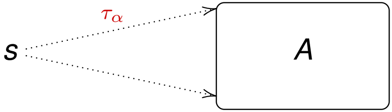
- For any subset of $\mathbf{R}$ we define outer measure as the infimum of the total length of the intervals of any covering family of intervals.
- The rationals have outer measure zero.
- This is not additive so it does not give a measure defined on all sets.
- It does however give a measure on the Borel sets.

Formal Definition of LMPs

An LMP is a tuple $(S, \Sigma, L, \forall \alpha \in L.\ \tau_\alpha)$

- (S, Σ) is an **analytic space**
- L is a countable set of labels
- $\tau_\alpha : S \times \Sigma \longrightarrow [0, 1]$ is a stochastic kernel, that is,

  - $\forall s \in S,\ \ \tau_\alpha(s, \cdot) : \Sigma \longrightarrow [0, 1]$ is a subprobability measure



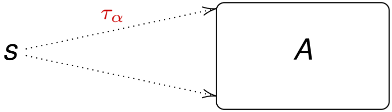
$\tau_\alpha(s, S) \leq 1$

Formal Definition of LMPs

An LMP is a tuple $(S, \Sigma, L, \forall\alpha \in L.\ \tau_\alpha)$

- (S, Σ) is an **analytic space**
- L is a countable set of labels
- $\tau_\alpha : S \times \Sigma \longrightarrow [0, 1]$ is a stochastic kernel, that is,
  - $\forall s \in S, \quad \tau_\alpha(s, \cdot\,) : \Sigma \longrightarrow [0, 1]$ is a subprobability measure
  - $\forall A \in \Sigma, \quad \tau_\alpha(\,\cdot, A) : S \longrightarrow [0, 1]$ is a measurable function.

  In particular, for $q \in \mathbb{Q}$: A $([q, 1])$



$\tau_\alpha(s, S) \leq 1$

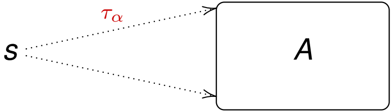
Formal Definition of LMPs

An LMP is a tuple $(S, \Sigma, L, \forall\alpha \in L.\ \tau_\alpha)$

- (S, Σ) is an **analytic space**
- L is a countable set of labels
- $\tau_\alpha : S \times \Sigma \longrightarrow [0, 1]$ is a stochastic kernel, that is,

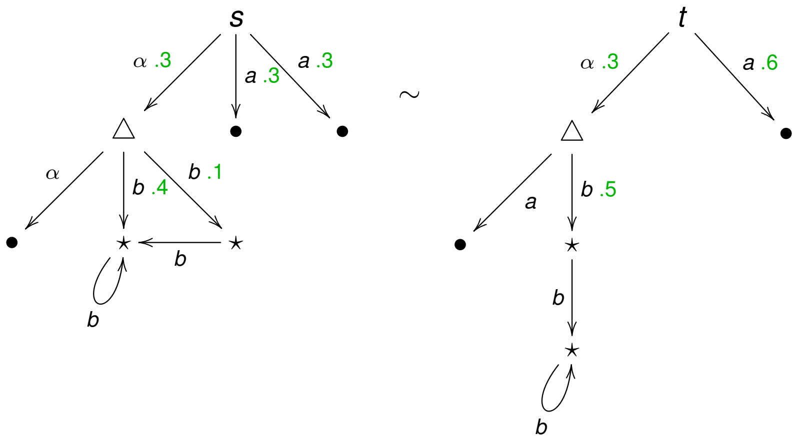
  - $\forall s \in S,\ \ \tau_\alpha(s, \cdot) : \Sigma \longrightarrow [0, 1]$ is a subprobability measure
  - $\forall A \in \Sigma,\ \ \tau_\alpha(\cdot, A) : S \longrightarrow [0, 1]$ is a measurable function.

    In particular, for $q \in \mathbb{Q}$: $\tau_\alpha(\cdot, A)^{-1}([q, 1]) \in \Sigma$



$\tau_\alpha(s, S) \leq 1$

Larsen-Skou Bisimulation - Example



Bisimulation

Let $\mathcal{S} = (S, i, \Sigma, \tau)$ a LMP and $R \subseteq S \times S$

A set is R-closed if whenever $s \in A$ and sRs' then $s' \in A$.

Bisimulation

Let $\mathcal{S} = (S, i, \Sigma, \tau)$ a LMP and $R \subseteq S \times S$

A set is R-closed if whenever $s \in A$ and sRs' then $s' \in A$.

Definition

An equivalence relation R is a bisimulation if

if $s\,R\,s'$, and if A is an R-closed set in Σ, then
$\tau_\alpha(s, A) = \tau_\alpha(s', A)$ for all $a \in$ L

s and t are bisimilar if sRt for some bisimulation relation.

Can be extended to bisimulation between two different **LMP**s.

Logic

$$\mathcal{L} ::== \mathsf{T} \mid \phi_1 \wedge \phi_2 \mid \langle \alpha \rangle_q \phi \qquad q \in \mathbb{Q} \cap [0,1]$$

$$s \models \langle \alpha \rangle_q \phi \quad \text{iff} \quad \tau_\alpha(s, [\![\phi]\!]) \geq q$$

where $[\![\phi]\!] := \{s \in S \mid s \models \phi\} \in \Sigma$

Logic

$$\mathcal{L} ::== \top \mid \phi_1 \wedge \phi_2 \mid \langle\alpha\rangle_q \phi \qquad q \in \mathbb{Q} \cap [0,1]$$

$$s \models \langle\alpha\rangle_q \phi \quad \text{iff} \quad \tau_\alpha(s, [\![\phi]\!]) \geq q$$

where $[\![\phi]\!] := \{s \in S \mid s \models \phi\} \in \Sigma$

Proof of $[\![\phi]\!] \in \Sigma$ by structural induction.

Base case: $[\![\top]\!] = S \in \Sigma$.

Inductive Step: let $[\![\phi_i]\!], [\![\phi]\!] \in \Sigma$

Logic

$$\mathcal{L} ::== \top \mid \phi_1 \wedge \phi_2 \mid \langle \alpha \rangle_q \phi \qquad q \in \mathbb{Q} \cap [0,1]$$

$$s \models \langle \alpha \rangle_q \phi \quad \text{iff} \quad \tau_\alpha(s, [\![\phi]\!]) \geq q$$

where $[\![\phi]\!] := \{s \in S \mid s \models \phi\} \in \Sigma$

Proof of $[\![\phi]\!] \in \Sigma$ by structural induction.

Base case: $[\![\top]\!] = S \in \Sigma$.
Inductive Step: let $[\![\phi_i]\!], [\![\phi]\!] \in \Sigma$

$[\![\phi_1 \wedge \phi_2]\!] = [\![\phi_1]\!] \cap [\![\phi_2]\!] \in \Sigma$.

Logic

$$\mathcal{L} ::== \mathrm{T} \mid \phi_1 \wedge \phi_2 \mid \langle\alpha\rangle_q \phi \qquad q \in \mathbb{Q} \cap [0,1]$$

$$s \models \langle\alpha\rangle_q \phi \quad \text{iff} \quad \tau_\alpha(s, [\![\phi]\!]) \geq q$$

where $[\![\phi]\!] := \{s \in S \mid s \models \phi\} \in \Sigma$

Proof of $[\![\phi]\!] \in \Sigma$ by structural induction.

Base case: $[\![\mathrm{T}]\!] = S \in \Sigma$.

Inductive Step: let $[\![\phi_i]\!], [\![\phi]\!] \in \Sigma$

$[\![\phi_1 \wedge \phi_2]\!] = [\![\phi_1]\!] \cap [\![\phi_2]\!] \in \Sigma$.

$[\![\langle a\rangle_q \phi]\!] = \{s \in S \mid \tau_\alpha(s, [\![\phi]\!]) \geq q\} =$

Logic

$$\mathcal{L} ::== \top \mid \phi_1 \wedge \phi_2 \mid \langle \alpha \rangle_q \phi \qquad q \in \mathbb{Q} \cap [0,1]$$

$$s \models \langle \alpha \rangle_q \phi \quad \text{iff} \quad \tau_\alpha(s, [\![\phi]\!]) \geq q$$

where $[\![\phi]\!] := \{s \in S \mid s \models \phi\} \in \Sigma$

Proof of $[\![\phi]\!] \in \Sigma$ by structural induction.

Base case: $[\![\top]\!] = S \in \Sigma$.
Inductive Step: let $[\![\phi_i]\!], [\![\phi]\!] \in \Sigma$

$[\![\phi_1 \wedge \phi_2]\!] = [\![\phi_1]\!] \cap [\![\phi_2]\!] \in \Sigma$.

$[\![\langle a \rangle_q \phi]\!] = \{s \in S \mid \tau_\alpha(s, [\![\phi]\!]) \geq q\} = \tau_\alpha(\cdot, [\![\phi]\!])^{-1}([q, 1]) \in \Sigma$

Logic

$$\mathcal{L} ::== \mathrm{T} \mid \phi_1 \wedge \phi_2 \mid \langle\alpha\rangle_q\phi \qquad q \in \mathbb{Q} \cap [0,1]$$

$$s \models \langle\alpha\rangle_q\phi \quad \text{iff} \quad \tau_\alpha(s, [\![\phi]\!]) \geq q$$

$$\text{where } [\![\phi]\!] := \{s \in S \mid s \models \phi\} \in \Sigma$$

Proof of $[\![\phi]\!] \in \Sigma$ by structural induction.

Base case: $[\![\mathrm{T}]\!] = S \in \Sigma$.
Inductive Step: let $[\![\phi_i]\!], [\![\phi]\!] \in \Sigma$

$[\![\phi_1 \wedge \phi_2]\!] = [\![\phi_1]\!] \cap [\![\phi_2]\!] \in \Sigma$.

$[\![\langle a\rangle_q\phi]\!] = \{s \in S \mid \tau_\alpha(s, [\![\phi]\!]) \geq q\} = \tau_\alpha(\cdot, [\![\phi]\!])^{-1}([q,1]) \in \Sigma$

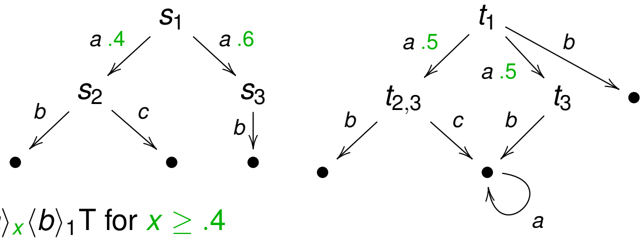
$[\![\neg\phi]\!] = [\![\phi]\!]^c \in \Sigma$ $\square$

Logic

$$\mathcal{L} ::== \top \mid \phi_1 \wedge \phi_2 \mid \langle \alpha \rangle_q \phi \qquad q \in \mathbb{Q} \cap [0,1]$$

$$s \models \langle \alpha \rangle_q \phi \quad \text{iff} \quad \tau_\alpha(s, [\![\phi]\!]) \geq q$$

where $[\![\phi]\!] := \{s \in S \mid s \models \phi\} \in \Sigma$



$s_1 \models \langle a \rangle_x \langle b \rangle_1 \top$ for $x \geq .4$

$\models \langle a \rangle_{.4}(\langle b \rangle_1 \top \wedge \langle c \rangle_1 \top)$

$t_1 \models \langle a \rangle_{.5} \langle b \rangle_1 \langle a \rangle_1 \langle a \rangle_1 \top$

Logic

$$\mathcal{L} ::== \mathsf{T} \mid \phi_1 \wedge \phi_2 \mid \langle\alpha\rangle_q \phi \qquad q \in \mathbb{Q} \cap [0,1]$$

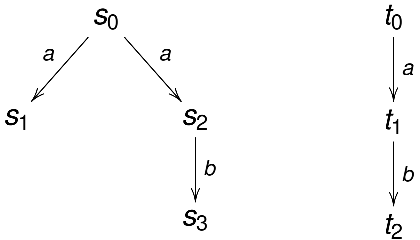
$$s \models \langle\alpha\rangle_q \phi \quad \text{iff} \quad \tau_\alpha(s, [\![\phi]\!]) \geq q$$

$$\text{where } [\![\phi]\!] := \{s \in S \mid s \models \phi\} \in \Sigma$$

Theorem (DEP, LICS 1998, I & C 2002)

Two systems with analytic state spaces are bisimilar iff they obey the same formulas of $\mathcal{L}$.

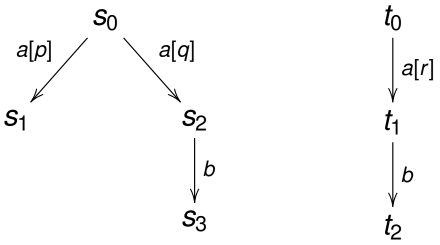
That cannot be right?



Two processes that cannot be distinguished without negation.
The formula that distinguishes them is $\langle a\rangle(\neg\langle b\rangle\top)$.

But it is!

We add probabilities to the transitions.



- If $p + q < r$ or $p + q > r$, then some $\langle a \rangle_x \top$ distinguishes them.
- If $p + q = r$ and $p > 0$ then $q < r$ so $\langle a \rangle_r \langle b \rangle_1 \top$ distinguishes them.

The Easy Direction: sRs' (bisimilar) $\Rightarrow s \sim_{\mathcal{L}} s'$

bisimulation R on (S, Σ, τ_α)
$s\,R\,s'$, A an R-closed set,
$\Rightarrow \tau_\alpha(s, A) = \tau_\alpha(s', A)$

$\mathcal{L}$: T | $\phi_1 \wedge \phi_2$ | $\langle\alpha\rangle_q\phi$
$s \models \langle\alpha\rangle_q\phi$
iff $\tau_\alpha(s, [\![\phi]\!]) \geq q$

We prove by induction on ϕ that $\forall \phi \in \mathcal{L}$

$[\![\phi]\!]$ is R-closed : i.e., $s \in [\![\phi]\!] \wedge sRs' \Rightarrow s' \in [\![\phi]\!]$

The Easy Direction: sRs' (bisimilar) $\Rightarrow s \sim_{\mathcal{L}} s'$

bisimulation R on (S, Σ, τ_α)
$s\,R\,s'$, A an R-closed set,
$\Rightarrow \tau_\alpha(s, A) = \tau_\alpha(s', A)$

$\mathcal{L}\ :\ \mathrm{T} \mid \phi_1 \wedge \phi_2 \mid \langle\alpha\rangle_q\phi$
$s \models \langle\alpha\rangle_q\phi$
iff $\tau_\alpha(s, [\![\phi]\!]) \geq q$

We prove by induction on ϕ that $\forall\phi \in \mathcal{L}$

$$[\![\phi]\!] \text{ is } R\text{-closed} : \text{ i.e., } s \in [\![\phi]\!] \wedge sRs' \Rightarrow s' \in [\![\phi]\!]$$

- Base case $[\![\mathrm{T}]\!] = S$
- $\wedge$ is obvious from Inductive Hypothesis.

The Easy Direction: sRs' (bisimilar) $\Rightarrow s \sim_{\mathcal{L}} s'$

bisimulation R on (S, Σ, τ_α)
$s\,R\,s'$, A an R-closed set,
$\Rightarrow \tau_\alpha(s, A) = \tau_\alpha(s', A)$

$\mathcal{L}\ :\ \mathrm{T} \mid \phi_1 \wedge \phi_2 \mid \langle\alpha\rangle_q\phi$
$s \models \langle\alpha\rangle_q\phi$
iff $\tau_\alpha(s, [\![\phi]\!]) \geq q$

We prove by induction on ϕ that $\forall\phi \in \mathcal{L}$

$$[\![\phi]\!] \text{ is } R\text{-closed}: \text{ i.e., } s \in [\![\phi]\!] \wedge sRs' \Rightarrow s' \in [\![\phi]\!]$$

- Base case $[\![\mathrm{T}]\!] = S$
- $\wedge$ is obvious from Inductive Hypothesis.
- $\phi = \langle a\rangle_q\psi$, where $[\![\psi]\!]$ R-closed from IH. Let $s \in [\![\phi]\!] \wedge sRs'$

The Easy Direction: sRs' (bisimilar) $\Rightarrow s \sim_{\mathcal{L}} s'$

bisimulation R on (S, Σ, τ_α)
$s\,R\,s'$, A an R-closed set,
$\Rightarrow \tau_\alpha(s, A) = \tau_\alpha(s', A)$

$\mathcal{L}\ : \mathrm{T} \mid \phi_1 \wedge \phi_2 \mid \langle\alpha\rangle_q \phi$
$s \models \langle\alpha\rangle_q \phi$
iff $\tau_\alpha(s, [\![\phi]\!]) \geq q$

We prove by induction on ϕ that $\forall \phi \in \mathcal{L}$

$$[\![\phi]\!] \text{ is } R\text{-closed} : \text{ i.e., } s \in [\![\phi]\!] \wedge sRs' \Rightarrow s' \in [\![\phi]\!]$$

- Base case $[\![\mathrm{T}]\!] = S$
- $\wedge$ is obvious from Inductive Hypothesis.
- $\phi = \langle a\rangle_q \psi$, where $[\![\psi]\!]$ R-closed from IH. Let $s \in [\![\phi]\!] \wedge sRs'$

$$\text{then } \tau_\alpha(s, [\![\psi]\!]) = \tau_\alpha(s', [\![\psi]\!])$$

thus $[\![\langle a\rangle_q \psi]\!]$ is R-closed. □

Proof Sketch of: s, s' bisimilar $\Leftarrow$ $s \sim_{\mathcal{L}} s'$

R is a bisimulation
$s\,R\,s'$, A an R-closed set,
$\Rightarrow \tau_\alpha(s, A) = \tau_\alpha(s', A)$

$\mathcal{L}$: $\mathrm{T} \mid \phi_1 \wedge \phi_2 \mid \langle\alpha\rangle_q\phi$
$s \models \langle\alpha\rangle_q\phi$
iff $\tau_\alpha(s, [\![\phi]\!]) \geq q$

Show that the relation $s \sim_{\mathcal{L}} s'$ is a bisimulation.

– this relation gives $\tau_\alpha(s, [\![\phi]\!]) = \tau_\alpha(s', [\![\phi]\!])$

Proof Sketch of: s, s' bisimilar $\Leftarrow$ $s \sim_{\mathcal{L}} s'$

R is a bisimulation
$s\,R\,s'$, A an R-closed set,
$\Rightarrow \tau_\alpha(s, A) = \tau_\alpha(s', A)$

$\mathcal{L}\ :\ \mathrm{T} \mid \phi_1 \wedge \phi_2 \mid \langle\alpha\rangle_q\phi$
$s \models \langle\alpha\rangle_q\phi$
iff $\tau_\alpha(s, [\![\phi]\!]) \geq q$

Show that the relation $s \sim_{\mathcal{L}} s'$ is a bisimulation.

- this relation gives $\tau_\alpha(s, [\![\phi]\!]) = \tau_\alpha(s', [\![\phi]\!])$
- $[\![\mathcal{L}]\!] := \{[\![\phi]\!] \mid \phi \in \mathcal{L}\}$ contains S, is countable and closed under intersection.
- $\tau_\alpha(s, \cdot)$ and $\tau_\alpha(s', \cdot)$ agree on $[\![\mathcal{L}]\!]$

Proof Sketch of: s, s' bisimilar $\Leftarrow$ $s \sim_{\mathcal{L}} s'$

Corollary (to Dynkin's λ-π theorem)

Two measures on (S, Σ) that agree on a π-system $\mathcal{F} \subseteq \Sigma$ (closed under $\cap$) agree on $\sigma(\mathcal{F})$.

Show that the relation $s \sim_{\mathcal{L}} s'$ is a bisimulation.

- this relation gives $\tau_\alpha(s, [\![\phi]\!]) = \tau_\alpha(s', [\![\phi]\!])$
- $[\![\mathcal{L}]\!] := \{[\![\phi]\!] \mid \phi \in \mathcal{L}\}$ contains S, is countable and closed under intersection.
- $\tau_\alpha(s, \cdot)$ and $\tau_\alpha(s', \cdot)$ agree on $[\![\mathcal{L}]\!]$

Proof Sketch of: s, s' bisimilar $\Leftarrow$ $s \sim_{\mathcal{L}} s'$

Theorem (1) (DP: JLAP03)

Let (S, Σ) be an analytic space, and $\mathcal{F} \subseteq \Sigma$ with $S \in \mathcal{F}$, countable and closed under intersection.
If two measures agree on $\mathcal{F}$, then they agree on $cl_\Sigma(\mathcal{F})$.

- this relation gives $\tau_\alpha(s, [\![\phi]\!]) = \tau_\alpha(s', [\![\phi]\!])$
- $[\![\mathcal{L}]\!] := \{[\![\phi]\!] \mid \phi \in \mathcal{L}\}$ contains S, is countable and closed under intersection.
- $\tau_\alpha(s, \cdot)$ and $\tau_\alpha(s', \cdot)$ agree on $[\![\mathcal{L}]\!]$

Proof Sketch of: s, s' bisimilar $\Leftarrow$ $s \sim_{\mathcal{L}} s'$

Theorem (1) (DP: JLAP03)

Let (S, Σ) be an analytic space, and $\mathcal{F} \subseteq \Sigma$ with $S \in \mathcal{F}$, countable and closed under intersection.
If two measures agree on $\mathcal{F}$, then they agree on $cl_{\Sigma}(\mathcal{F})$.

- this relation gives $\tau_\alpha(s, [\![\phi]\!]) = \tau_\alpha(s', [\![\phi]\!])$
- $[\![\mathcal{L}]\!] := \{[\![\phi]\!] \mid \phi \in \mathcal{L}\}$ contains S, is countable and closed under intersection.
- $\tau_\alpha(s, \cdot)$ and $\tau_\alpha(s', \cdot)$ agree on $[\![\mathcal{L}]\!]$
- by Theorem (1), if (S, Σ) is analytic, they agree on $cl_{\Sigma}([\![\mathcal{L}]\!])$

Proof Sketch of: s, s' bisimilar $\Leftarrow$ $s \sim_{\mathcal{L}} s'$

R is a bisimulation
$s\, R\, s'$, A an R-closed set,
$\Rightarrow \tau_\alpha(s, A) = \tau_\alpha(s', A)$

$\mathcal{L}$: $\mathrm{T} \mid \phi_1 \wedge \phi_2 \mid \langle\alpha\rangle_q\phi$
$s \models \langle\alpha\rangle_q\phi$
iff $\tau_\alpha(s, [\![\phi]\!]) \geq q$

Show that the relation $s \sim_{\mathcal{L}} s'$ is a bisimulation.

- this relation gives $\tau_\alpha(s, [\![\phi]\!]) = \tau_\alpha(s', [\![\phi]\!])$
- $[\![\mathcal{L}]\!] := \{[\![\phi]\!] \mid \phi \in \mathcal{L}\}$ contains S, is countable and closed under intersection.
- $\tau_\alpha(s, \cdot)$ and $\tau_\alpha(s', \cdot)$ agree on $[\![\mathcal{L}]\!]$
- by Theorem (1), if (S, Σ) is analytic, they agree on $cl_\Sigma([\![\mathcal{L}]\!])$
- $\sim_{\mathcal{L}}$-closed sets are exactly members of $cl_\Sigma([\![\mathcal{L}]\!])$. $\square$

Hence negation plays no role!

Digression on Analytic Spaces

The last step of the previous proof used:

Theorem (1) (DP: JLAP03)

Let (S, Σ) be an analytic space, and $\mathcal{F} \subseteq \Sigma$ with $S \in \mathcal{F}$, countable and closed under intersection.
If two measures agree on $\mathcal{F}$, then they agree on $cl_{\Sigma}(\mathcal{F})$.

The first step is the following theorem

Corollary (to Dynkin's λ-π theorem)

Two measures that agree on a π-system $\mathcal{F}$ agree on $\sigma(\mathcal{F})$.

Digression on Analytic Spaces

The last step of the previous proof used:

Theorem (1) (DP: JLAP03)

Let (S, Σ) be an analytic space, and $\mathcal{F} \subseteq \Sigma$ with $S \in \mathcal{F}$, countable and closed under intersection.
If two measures agree on $\mathcal{F}$, then they agree on $cl_{\Sigma}(\mathcal{F})$.

The first step is the following theorem

Corollary (to Dynkin's λ-π theorem)

Two measures that agree on a π-system $\mathcal{F}$ agree on $\sigma(\mathcal{F})$.

Now look at the following theorem on analytic spaces:

Theorem (Unique Structure Theorem)

If (S, Σ) is an analytic space, Σ_0 a sub-σ-algebra of Σ that separates points and is countably generated then $\Sigma_0 = \Sigma$.

Analytic Spaces

Definition

An analytic set A is the image of a Polish space X (or a Borel subset of X) under a continuous (or measurable) function $f : X \rightarrow Y$, where Y is Polish.

Analytic Spaces

Definition

An analytic set A is the image of a Polish space X (or a Borel subset of X) under a continuous (or measurable) function $f : X \longrightarrow Y$, where Y is Polish.

Theorem (quotient of analytic is analytic)

Given (S, Σ) an analytic space and $\sim$ an equivalence relation such that there is a countable family of real-valued measurable functions $f_i : S \longrightarrow \mathbf{R}$ such that

$$\forall s, s' \in S . s \sim s' \iff \forall f_i \,.\; f_i(s) = f_i(s')$$

then the quotient space (Q, Ω) - where $Q = S/\sim$ and Ω is the finest σ-algebra making the canonical surjection $q : S \longrightarrow Q$ measurable - is also analytic.

The Quotient

Theorem (unique measure)

Let (S, Σ) be an analytic space, and $\mathcal{F} \subseteq \Sigma$ with $S \in \mathcal{F}$, countable and closed under intersection.
If two measures agree on $\mathcal{F}$, then they agree on $cl_\Sigma(\mathcal{F})$.

$cl_\Sigma(\mathcal{F}) := \{A \in \Sigma \mid \text{ if } s \in A \text{ and } s \equiv_\mathcal{F} s' \text{ then } s' \in A\}$
The equivalence $s \equiv_\mathcal{F} s'$ is witnessed also by the functions $I_F : S \longrightarrow \mathbf{R}$, for $F \in \mathcal{F}$ defined by

$$I_F(s) = 1 \text{ if } s \in F, \text{ and 0 otherwise}$$

They are a countable family of measurable functions.
Thus the quotient space (Q, Ω) is analytic.

Through the quotient: $q : (S, \Sigma) \longrightarrow (Q, \Omega)$

Recall that $\Omega := \{Y \subseteq Q \mid q^{-1}(Y) \in \Sigma\}$

We prove $q(cl_\Sigma(\mathcal{F})) = \Omega$

- $\supseteq$: because $q^{-1}(Y)$ is $\equiv_\mathcal{F}$-closed for $Y \in \Omega$
- $\subseteq$: if $X \in cl_\Sigma(\mathcal{F})$ then $q(X) \in \Omega$ because $q^{-1}(q(X)) = X$ (1)

  $s \in q^{-1}(q(X))$ implies that $q(s) \in q(X)$, i.e. $\exists s' \in X.s \simeq s'$, but X is closed so $s \in X$.

Now $q(\sigma(\mathcal{F}))$

- is a sub-σ-algebra of Ω (inclusion is by (1))
- $= \sigma(q(\mathcal{F}))$ and hence is countably generated and separates points

Thus $q(\sigma(\mathcal{F})) = q(cl_\Sigma(\mathcal{F}))$

The argument finishes with $\sigma(\mathcal{F}) = cl_\Sigma(\mathcal{F})$ □

Simulation on an LMP $\mathcal{S} = (S, \Sigma, \tau)$

Definition (DGJP I&C03)

A preorder R is a simulation if

if $s\,R\,s'$, and if A is an R-closed set in Σ, then
$\tau_\alpha(s, A) \leq \tau_\alpha(s', A)$ for all $a \in$ L

s and t are bisimilar if sRt for some bisimulation relation.

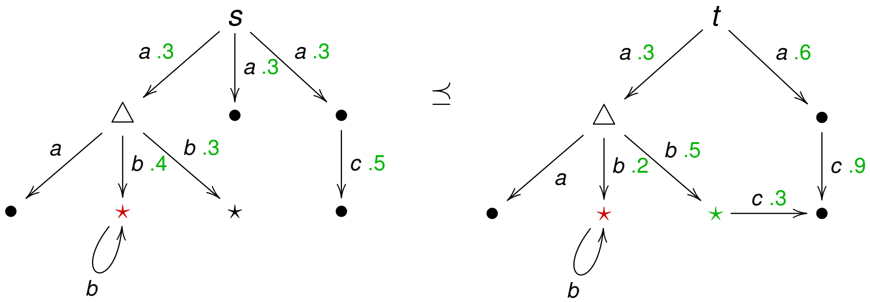
Simulation on an LMP $\mathcal{S} = (S, \Sigma, \tau)$

Definition (DGJP I&C03)

A preorder R is a simulation if

if $s \, R \, s'$, and if A is an R-closed set in Σ, then
$\tau_\alpha(s, A) \leq \tau_\alpha(s', A)$ for all $a \in$ L

s and t are bisimilar if sRt for some bisimulation relation.



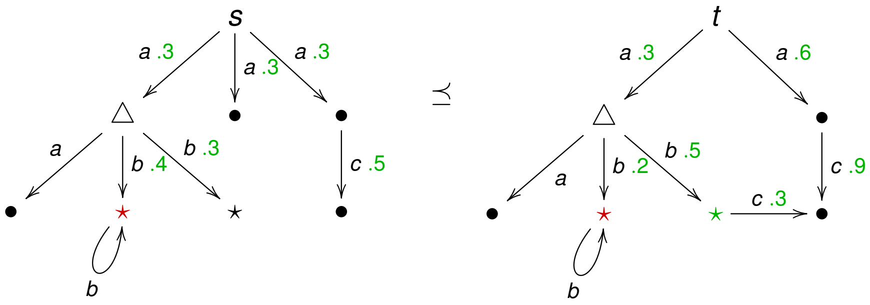
Simulation on an LMP $\mathcal{S} = (S, \Sigma, \tau)$

Definition (DLT in QEST08)

A preorder R is a ϵ-simulation if

if $s\,R\,s'$, and if A is an R-closed set in Σ, then
$$\tau_\alpha(s, A) \leq \tau_\alpha(s', A) - \epsilon \quad \text{for all } a \in L$$

s and t are bisimilar if sRt for some bisimulation relation.

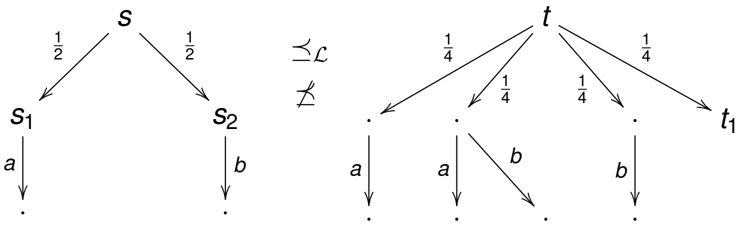


Logic for simulation?

- The logic used in the characterization has no negation, not even a limited negative construct.
- One can show that if s simulates s' then s satisfies all the formulas of $\mathcal{L}$ that s' satisfies.
- What about the converse?

Counter example!

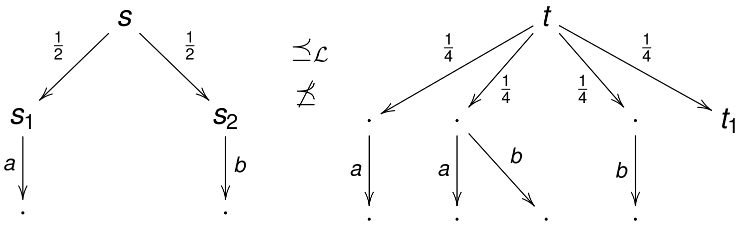
In the following picture, t satisfies all formulas of $\mathcal{L}$ that s satisfies but t does not simulate s.



All transitions from s and t are labelled by a.

Counter example!

In the following picture, t satisfies all formulas of $\mathcal{L}$ that s satisfies but t does not simulate s.



All transitions from s and t are labelled by a.

t_1 cannot simulate any state but t reaches it with probability $\frac{1}{4}$

$s \models \langle a \rangle_{\frac{7}{8}}(\langle a \rangle_{0.1}\mathrm{T} \vee \langle b \rangle_{0.1}\mathrm{T})$
$t \not\models$

$t \models \langle a \rangle_{0.1}(\langle a \rangle_{0.1}\mathrm{T} \wedge \langle b \rangle_{0.1}\mathrm{T})$.
$s \not\models$

so $s \not\prec_{\mathcal{L}} t$

A logical characterization for simulation

The logic $\mathcal{L}$ does **not** characterize simulation. One needs disjunction.

$$\mathcal{L}_\vee := \mathcal{L} \mid \phi_1 \vee \phi_2.$$

Theorem (DGJP I&C03)

*An **LMP** s_1 simulates s_2 if and only if for every formula ϕ of $\mathcal{L}_\vee$ we have*

$$s_1 \models \phi \Rightarrow s_2 \models \phi.$$

The only proof we know uses domain theory.

Other Logics

$$\begin{aligned}
\mathcal{L}_{\mathrm{Can}} &:= \mathcal{L}_0 \mid \mathrm{Can}(a) \\
\mathcal{L}_{\Delta} &:= \mathcal{L}_0 \mid \Delta_a \\
\mathcal{L}_{\neg} &:= \mathcal{L}_0 \mid \neg\phi \\
\mathcal{L}_{\wedge} &:= \mathcal{L}_{\neg} \mid \bigwedge_{i \in \mathbf{N}} \phi_i
\end{aligned}$$

where

$$\begin{aligned}
s \models \mathrm{Can}(a) &\quad \text{to mean that } \tau_a(s, S) > 0; \\
s \models \Delta_a &\quad \text{to mean that } \tau_a(s, S) = 0.
\end{aligned}$$

We need $\mathcal{L}_{\vee}$ to characterise simulation.

Conclusions

- Strong probabilistic bisimulation is characterised by a very simple modal logic with no negative constructs.
- There is a logical characterisation of simulation.
- There is a “metric” on LMPs which is based on this logic.
- Why did the proof require so many subtle properties of analytic spaces? The logical characterisation proof is “easy” for event- bisimulation, but the two bisimulations coincide only on analytic spaces.