On a Categorical Framework for Coalgebraic Modal Logic

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Motivation

I was looking for a structural approach to coalgebraic modal logic independent of Stone duality, maps between semantics, and . . . a category!

Modularity	Colimits, limits, and compositions. No syntax bookkeeping!
Full semantics	The terminal object in some fibre.
Modality	Objects of modalities characterised by Yoneda Lemma.
Generality	None of results depends on any particular propositional logic.

What is . . . *coalgebraic modal logic*?

Modalities for coalgebras for some $T\colon \mathbf{Set} \to \mathbf{Set}$ can be modelled in different ways:

1. cover modality $\nabla\colon T2^{-} \Rightarrow 2^{T}$ by Moss for weak-pullback preserving T (not discussed here).
2. n-ary predicate liftings by Pattinson $\lambda : (2^{-})^{n} \Rightarrow (2^{T})$

Definition (Logic of predicate liftings)

Syntax $\Phi \ni \varphi := \bot \mid \neg\varphi \mid \varphi \wedge \varphi \mid \overline{\lambda}(\varphi_i)_{i=1\ldots n}$.

Semantics For $(X \xrightarrow{\xi} TX)$, define $[\![-]\!]\colon \Phi \to 2^{X}$ by $[\![\bot]\!] := \emptyset$, $[\![\neg\varphi]\!] := [\![\varphi]\!]^{\complement}$, $[\![\varphi \wedge \psi]\!] := [\![\varphi]\!] \cap [\![\psi]\!]$, and

$$[\![\overline{\lambda}(\varphi_i)]\!] := \xi^{-1} \circ \lambda_X([\![\varphi_i]\!])$$

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Example: Kripke semantics via predicate lifting

1. Classical modal logic

$$\varphi := \bot \mid \neg\varphi \mid \varphi \wedge \varphi \mid \Diamond\varphi$$

where $\Diamond$ is a predicate lifting for $\mathbb{P}$ defined by

$$\Diamond_X \colon (S \subseteq X) \mapsto \{\, U \in \mathbb{P}X \mid U \cap S \neq \emptyset \,\}.$$

2. For any $x \in (X, \xi)$, we have

$$\begin{aligned} & x \in [\![\Diamond\varphi]\!] = \xi^{-1} \circ \Diamond_X([\![\varphi]\!]) \\ \iff & x \in \{\, x \in X \mid \xi(x) \cap [\![\varphi]\!] \neq \emptyset \,\} \\ \iff & \exists y \in \xi(x).\, y \in [\![\varphi]\!] \end{aligned}$$

That is, the usual semantics of possibility.

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One-step semantics for Stone duality

Facts (Kupke, Kurz and Pattinson, 2004) and (Kurz and Leal, 2012)

Both of approaches for T-coalgebras are of the following form

$$L\mathcal{Q} \stackrel{\cdot}{\longrightarrow} \mathcal{Q}T$$

where $\mathcal{Q}$: **Set** $\to$ **BA** is the contravariant powerset algebra functor.

Example (L as presentation of modalities)

Define $L = \mathbb{M}$: **BA** $\to$ **BA** by the presentation

$$\mathbb{M}A := \mathrm{BA}\langle\, \{\blacklozenge a\}_{a \in A} \mid \blacklozenge\bot = \bot, \blacklozenge(a \vee b) = \blacklozenge a \vee \blacklozenge b \,\rangle$$

and $(\mathbb{M}f)(\blacklozenge a) := \blacklozenge fa$. Every Boolean algebra with a join-preserving function $\Diamond$ is an $\mathbb{M}$-algebra $\alpha\colon \mathbb{M}A \to A$ by

$$\alpha(\blacklozenge a) := \Diamond(a) \quad \text{and conversely} \quad \Diamond(a) := \alpha(\blacklozenge a).$$

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1. For each set X, define $\delta_X \colon \mathbb{M}\mathcal{Q}X \xrightarrow{\cdot} \mathcal{Q}\mathbb{P}X$ on $\blacklozenge S$ on generators by
$$\blacklozenge S \mapsto \Diamond_X(S)$$
where $S \subseteq X$.
2. Every $\mathbb{P}$-coalgebra is mapped to an $\mathbb{M}$-algebra by
$$\mathcal{Q}^\delta \colon (X \xrightarrow{\xi} \mathbb{P}X) \mapsto (\mathbb{M}\mathcal{Q}X \xrightarrow{\delta_X} \mathcal{Q}\mathbb{P}X \xrightarrow{\mathcal{Q}\xi} \mathcal{Q}X)$$
which is the complex algebra.
3. The interpretation for (X, ξ) is unique morphism from the initial $\mathbb{M}$-algebra (Φ, α):

$$\begin{array}{ccccc}
\mathbb{M}\Phi & & \xrightarrow[\cong]{\alpha} & & \Phi \\
{\scriptstyle \mathbb{M}[\![-]\!]}\downarrow & & & & \downarrow{\scriptstyle [\![-]\!]} \\
\mathbb{M}\mathcal{Q}X & \xrightarrow[\delta_X]{} & \mathcal{Q}\mathbb{P}X & \xrightarrow[\mathcal{Q}\xi]{} & \mathcal{Q}X
\end{array}$$

E.g. $[\![\alpha(\blacklozenge\varphi)]\!] = (\mathcal{Q}\xi \circ \delta_X \circ \mathbb{M}[\![-]\!])(\blacklozenge\varphi) = (\xi^{-1} \circ \Diamond_X)([\![\varphi]\!])$.

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One-step semantics in general

Definition

A **one-step semantics** over a contravariant functor $P\colon \mathscr{X} \to \mathscr{A}$ consists of

$$
\begin{aligned}
\text{type of behaviour} \quad & T\colon \mathscr{X} \to \mathscr{X}, \\
\text{syntax of modalities} \quad & L\colon \mathscr{A} \to \mathscr{A}, \text{ and} \\
\text{interpretation of modalities} \quad & \delta\colon LP \stackrel{\cdot}{\longrightarrow} PT.
\end{aligned}
$$

$P\colon \mathscr{X} \to \mathscr{A}$ usually forms a dual adjuntion on the right with some S, i.e.

$$\mathscr{X}(X, SA) \cong \mathscr{A}(A, PX)$$

natural in X and A.

Predicates

Example

1. Stone dualities $\mathcal{Q}\colon \mathbf{Set} \to \mathbf{BA}$, $2^{-}\colon \mathbf{Set} \to \mathbf{Set}$, $\mathcal{O}\colon \mathbf{Top} \to \mathbf{Frm}$, and $\mathrm{Clp}\colon \mathbf{Stone} \to \mathbf{BA}$.
2. $\mathbb{S}\colon \mathbf{Meas} \to \wedge\text{-}\mathbf{SLat}$ maps a measurable space to its σ-algebra as a $\wedge$-semilattice.

- Let P be one of the above with the dual adjoint S;
- $\mathscr{A} = \mathbf{Set}, \mathbf{BA}, \mathbf{Frm}, \wedge\text{-}\mathbf{SLat}$ with F the left adjoint to the forgetful functor $U\colon \mathscr{A} \to \mathbf{Set}$.
- UPX is understood as "predicates" on X and moreover

$$UPX \cong \mathbf{Set}(1, UPX) \cong \mathscr{A}(F1, PX) \cong \mathscr{X}(X, SF1)$$

- A predicate on X is a test on X by Ω.

- for $2^{-}\colon \textbf{Set} \to \textbf{Set}$ and $\mathcal{Q}\colon \textbf{Set} \to \textbf{BA}$,

$$\Omega \cong \{\bot, \top\}$$

- for $\mathcal{O}\colon \textbf{Top} \to \textbf{Frm}$,

$$\Omega \cong (\{\bot, \top\}, \{\emptyset, \{\top\}, \{\bot, \top\})$$

  the Sierpiński space.

- for the clopen funtor $\mathrm{Clp}\colon \textbf{Stone} \to \textbf{BA}$ and $\mathbb{S}\colon \textbf{Meas} \to \wedge\text{-}\textbf{SLat}$,

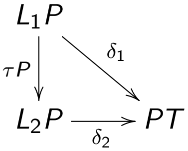
$$\Omega \cong (\{\bot, \top\}, \{\emptyset, \{\top\}, \{\bot\}, 2\}\}$$

  the discrete space on $\{\bot, \top\}$.

Notions of maps between semantics

Question: What are morphisms between one-step semantics over the same P? Two possible choices

1. Interpretation-preserving translations $\tau \colon L_1 \stackrel{\cdot}{\longrightarrow} L_2$ between syntaxes



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$$\begin{array}{ccc} L_1 P & & \\ \downarrow{\scriptstyle \tau P} & \searrow^{\delta_1} & \\ L_2 P & \xrightarrow[\delta_2]{} & PT \end{array}$$

2. Natural transformations between types of behaviour $\nu\colon T_2 \to T_1$ satisfying

$$\begin{array}{ccc} LP & \xrightarrow{\delta_1} & PT_1 \\ & {}_{\delta_2}\searrow & \downarrow{\scriptstyle P\nu} \\ & & PT_2 \end{array}$$

My answer: Both. A morphism from (L_1, T_1, δ_1) to (L_2, T_2, δ_2) is a pair of natural transformations $\tau \colon L_1 \to L_2$ and $\nu \colon T_2 \to T_1$ satisfying

$$\begin{array}{ccc} L_1 P & \xrightarrow{\delta_1} & P T_1 \\ {\scriptstyle \tau P}\downarrow & & \downarrow{\scriptstyle P\nu} \\ L_2 P & \xrightarrow[\delta_2]{} & P T_2 \end{array}$$

The previous choices are special cases for $\nu = id$ or $\tau = id$ respectively.

Category of One-Step Semantics

The **category of one-step semantics** over P, denoted **CoLog** (with P implicit) is a category consisting of

objects one-step semantics $(L, T, \delta\colon LP \overset{\cdot}{\longrightarrow} PT)$ over P.

morphisms a pair $(\tau\colon L_1 \overset{\cdot}{\longrightarrow} L_2, \nu\colon T_2 \overset{\cdot}{\longrightarrow} T_1)$ of nat. trans. is a morphism from (L_1, T_1, δ_1) to (L_2, T_2, δ_2) if

$$\begin{array}{ccc} L_1 P & \xrightarrow{\delta_1} & PT_1 \\ {\scriptstyle \tau P}\downarrow & & \downarrow{\scriptstyle P\nu} \\ L_2 P & \xrightarrow[\delta_2]{} & PT_2 \end{array}$$

In short, **CoLog** is the comma category $(P^* {\downarrow} P_*)$ from the pre-composition of P to the post-composition P.

Modularity: Colimits of one-step semantics

If pointwise coproduct L_1 and L_2 exists, then

$$
\begin{array}{ccc}
L_1 P & \xrightarrow{\delta_1} & PT_1 \\
\downarrow{\scriptstyle inj_1 P} & & \downarrow{\scriptstyle P\,proj_1} \\
(L_1 + L_2)P & \overset{\delta}{\dashrightarrow} & P(T_1 \times T_2) \\
\uparrow{\scriptstyle inj_2 P} & & \uparrow{\scriptstyle P\,proj_2} \\
L_2 P & \xrightarrow[\delta_2]{} & PT_2
\end{array}
$$

and δ is a coproduct in **CoLog**. It also applies to colimits in general.

Example (Labelling T^A)

The A-fold coproduct of modal logic is multi-modal logic for A-labelled Kripke frames $X \to (\mathbb{P}X)^A$.

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Modularity: Product of one-step semantics

Assume P has a dual adjoint. Then,

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\begin{array}{ccc}
L_1 P & \xrightarrow{\delta_1} & PT_1 \\
{\scriptstyle proj_1 P} \uparrow & & \uparrow {\scriptstyle Pinj_1} \\
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and δ is a product in **CoLog**. It applies to limits in general.

Example

1. An **alternating system** over an action set A is a coalgebra for $\mathcal{D} + \mathbb{P}^A$ where $\mathcal{D}$ is the probability distribution functor.
2. A modal logic for alternating system is a product of probabilistic modal logic and A-labelled modal logic.

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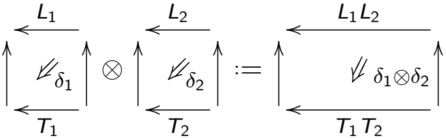
Modularity: Compositions of one-step semantics

Endofunctors are composable, so are one-step semantics. Define the composition $\delta_1 \otimes \delta_2$ of δ_1 and δ_2 by *pasting diagrams*



that is,

$$\delta_1 \otimes \delta_2 \colon L_1 L_2 P \xrightarrow{L_1 \delta_2} L_1 P T_2 \xrightarrow{\delta_1 T_2} P T_1 T_2.$$

Theorem

The composition $\otimes$ with the identity semantics $(\mathcal{I}, \mathcal{I}, id_P)$ is a strict monoidal structure on **CoLog**.

Example: Simple Segala system

Definition

A **simple Segala system** for a set A of actions is a

1. a coalgebra for $\mathbb{P}^A \circ \mathcal{D}$
2. $\mathcal{D}$ is the probability distribution functor defined by

$$\mathcal{D}X := \{\, \mu : X \to [0,1] \mid \sum_{x \in X} \mu(x) = 1 \text{ and } |\mu(x) \neq 0| \in \mathbb{N} \,\}$$

for each set X.

A modal logic for simple Segala systems can be derived as the composition of

1. the A-fold coproduct $\coprod_A(\mathbb{M}, \mathbb{P}, \delta)$ and
2. probabilistic modal logic $(L^\Lambda, \mathcal{D}, \delta^\Lambda)$ induced by predicate liftings

$$\langle p \rangle(S) := \{\, \mu \in \mathcal{D}X \mid \sum \mu(S) \geq p \,\}$$

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Monoid objects

Theorem

*A **monoid object** in the strict monoidal category* $(\mathbf{CoLog}, \otimes, id)$ *consists of a monad* $\mathbb{L}$ *on* $\mathscr{A}$*, a comonad* $\mathbb{T}$ *on* $\mathscr{X}$ *and a one-step semantics* $\delta : \mathbb{L}P \to P\mathbb{T}$ *satisfying the **homomorphism** condition.*

Are they multi-step semantics? Any other kind of objects? Any use? Future work.

Remark

Some of them were done in **Set** by (Cîrstea and Pattinson, 2007) and (Schröder and Pattinson, 2011).

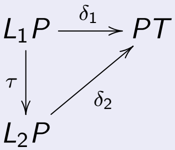
Category of semantics for T-coalgebras

Now, we fix the type of behaviour ...

Definition

The category $\mathbf{CoLog}_T$ of one-step semantics for T-coalgebras consists of

- objects: natural transformations $\delta\colon LP \stackrel{\cdot}{\longrightarrow} PT$, i.e. one-step semantics (L, δ) with T implicit.
- morphisms: a natural transformation $\tau\colon L_1 \to L_2$ is a morphism from (L_1, T, δ_1) to (L_2, T, δ_2) if

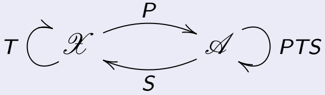


In short, $\mathbf{CoLog}_T$ is the fibre over T.

Terminal object in $\mathbf{CoLog}_T$

Theorem

Suppose that P has a dual adjoint S. Then every fibre CoLog_T has a terminal object $(PTS, PT\epsilon\colon PTSP \stackrel{\cdot}{\longrightarrow} PT)$



where $\epsilon\colon \mathcal{I} \to SP$ is the counit of the dual adjunction.

- This one-step semantics is called the **full one-step semantics** for T-coalgebras.
- Its presentation is not clear, but we will restrict to its equationally presentable part shortly.

1. For every (L, δ), there is $\tau\colon L \to PTS$ by $L \xrightarrow{L\eta} LPS \xrightarrow{\delta S} PTS$.
2. τ is a translation because

3. for every translation $\tau'\colon (L, \delta) \to (PTS, PT\epsilon)$ the diagram

commutes, so $\tau = \tau'$.

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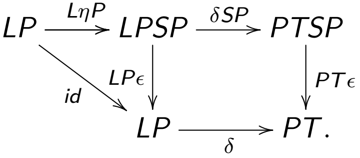

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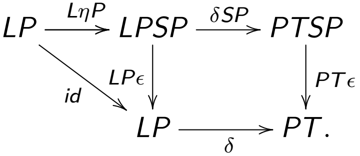

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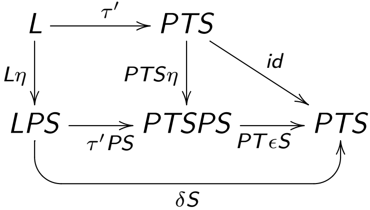


commutes, so $\tau = \tau'$.

Finitely based functors

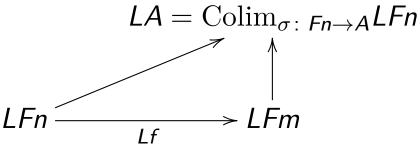
We restrict to those "presentable" L:

Definition (see (Bonsangue and Kurz, 2006) and (Velebil and Kurz, 2011))

Let $U\colon \mathscr{A} \to \mathbf{Set}$ be a finitary and monadic functor. A functor $L\colon \mathscr{A} \to \mathscr{A}$ is **finitely based** if one of the following holds:

1. L is finitary and preserves canonical presentations;
2. L preserves sifted colimtis;
3. L is a left Kan extension $\mathrm{Lan}_J LJ$ for the inclusion function from the subcategory of $\mathscr{A}$ on Fn for $n \in \mathbb{N}$.

That is,



Fact

Every finitely based functor $L \colon \mathscr{A} \to \mathscr{A}$ is isomorphic to a functor defined by

$$LA \cong \mathscr{A}\langle \{\sigma(\vec{a})\}_{\sigma\in\Sigma_n,\vec{a}\in A^n} \mid \mathcal{E} \rangle$$

where Σ_n is a set of n-ary operations and $\mathcal{E}$ a set of rank-1 equations.

E.g. $\mathbb{M}$, L^Λ for a set Λ of predicate liftings.

Theorem

The category $\mathbf{Fin}[\mathscr{A},\mathscr{A}]$ of finitely based endofunctors is the coreflexive subcategory of the category $[\mathscr{A},\mathscr{A}]$ of endofunctors.

A finitely based coreflection ρ of L is a natural transformation

$$\rho_L \colon \mathrm{Lan}_J LJ \overset{\cdot}{\longrightarrow} L$$

derived by $\mathrm{Colim}_{Fn\to A} LFn$.

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Categories for equational semantics

Assume that there is a finitary and monadic $U\colon \mathscr{A} \to \mathbf{Set}$.

Definition

1. The category **ECoLog** is the subcategory of **CoLog** on one-step semantics whose syntax functor L is *finitely based*.
2. The category $\mathbf{ECoLog}_T$ is the subcategory of $\mathbf{CoLog}_T$ and a fibre of **ECoLog** over T.

Proposition

1. **ECoLog** is a coreflexive subcategory of **CoLog**.
2. $\mathbf{ECoLog}_T$ is a coreflexive subcategory of $\mathbf{CoLog}_T$.

The coreflection is derived by precomposing the coreflection ρ_L of L

$$(\mathrm{Lan}_J LJ)P \stackrel{\cdot}{\longrightarrow} LP \stackrel{\cdot}{\longrightarrow} PT$$

Moduarity, revisited

Proposition

ECoLog is closed under colimits, finite products, and compositions.

- It follows from coreflexivity, the commuting property of sifted colimits with finite products, and the preservation property.
- E.g. A-fold coproduct of $\mathbb{M}$

$$\coprod_A \mathbb{M}B \cong \mathbf{BA}\langle \{\Diamond_i a\}_{i\in A, b\in B} \mid \cdots \rangle$$

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Terminal object in $\mathbf{ECoLog}_T$

Theorem

Suppose that P has a dual adjoint S. Then every fibre ECoLog_T *has a terminal object*

$$(\mathrm{Lan}_J PTSJ, PT\epsilon \circ \rho P \colon (\mathrm{Lan}_J PTSJ)P \dot{\longrightarrow} PT)$$

where J is the inclusion from the subcategory of A on Fn for $n \in \mathbb{N}$, $\epsilon \colon \mathcal{I} \to SP$ is the counit of the dual adjunction and $\rho = \rho_{PTS}$ is the finitely based coreflection of PTS.

- This one-step semantics is called the **full equational one-step semantics** for T-coalgebras.
- It corresponds to the **logic of all finitary predicate liftings** subject to a **complete axiomatisation**.[1]

[1] Its characterisation is ignored in this talk, see my MFPS paper or thesis.

Object of predicate liftings

Let F be the left adjoint to $U\colon \mathscr{A} \to \mathbf{Set}$. Note that $\mathrm{Lan}_J PTSJ$ can be computed as $\mathrm{Colim}_{Fn \to A} PTSFn$ on $A \in \mathscr{A}$.

Lemma

For each $n \in \mathbb{N}$, there is a natural isomorphism

$$UPTSFn \cong \mathbf{Nat}(UP^n, UPT)$$

- For $P = 2^-, \mathcal{Q}$, a natural transformation from UP^n to UPT coincides with a predicate lifting for a set functor T.
- We shall call a natural transformation $\lambda\colon UP^n \to UPT$ a **predicate lifting** either.

Proof.

By Yoneda Lemma, the dual adjunction, and the free adjunction, we have

$$\begin{aligned} UPTSFn &\cong \mathbf{Nat}(\mathscr{X}(-, SFn), UPT) \\ &\cong \mathbf{Nat}(\mathscr{A}(Fn, P-), UPT) \\ &\cong \mathbf{Nat}(UP^n, UPT) \end{aligned}$$

□

Remark

1. It is known in (Schröder, 2008) for 2^- by Yoneda Lemma and the fact that $2^- \cong \mathbf{Set}(-, 2)$ is representable.
2. It was suggested implicitly in (Klin, 2007).

A higher generality gives an even simpler argument.

Theory map

For simplicity, assume that the category $\mathscr{X}$ of state spaces is concrete.

1. Let (L, T, δ) be a one-step semantics such that the initial L-algebra (Φ, α) exists.
2. The interpretation $[\![-]\!]\colon \Phi \to PX$ for a T-coalgebra (X, ξ) is the unique L-algebra homomorphism to $(LPX \xrightarrow{\delta_X} PTX \xrightarrow{P\xi} PX)$ as usual.
3. The **theory map** $th\colon X \to S\Phi$ is the transpose of $[\![-]\!]$

E.g. for $P = 2^{-}$, the theory of x is the set of true propositions on x

$$th(x) = \{\, \varphi \in \Phi \mid x \in [\![\varphi]\!] \,\}$$

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Logical equivalence, adequacy, and expressiveness

1. Two elements x and y are **logically equivalent** wrt (L, T, δ) if $th(x) = th(y)$.

2. x and y are **behaviourally equivalent** (or bisimilar) wrt T if there exists a coalgebra homomorphism f with $f(x) = f(y)$.

3. A logic is **adequate** if behaviourally equivalent elements are logically equivalent. It holds for all one-step semantics (L, T, δ) with an initial L-algebra.

4. A logic is **expressive** if every two logically equivalent elements are behaviourally equivalent.

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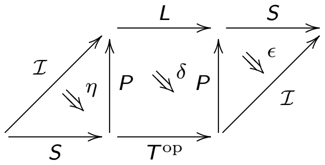
One-step expressiveness

Define the **mate** δ^* for $\delta\colon LP \overset{\cdot}{\longrightarrow} PT$ by pasting



i.e. δ^* is a natural transformation $SL\eta \circ S\delta S \circ \epsilon TS$ from TS to SL.

Theorem ((Klin, 2007) and (Jacobs and Sokolova, 2010))

Suppose that $\mathscr{X}$ has a proper factorisation system $(\mathcal{E}, \mathcal{M})$. Then, a one-step semantics (L, T, δ) is expressive if

1. *T preserves $\mathcal{M}$-morphisms and*
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A one-step semantics is **one-step expressive** if it satisfies the above two conditions.

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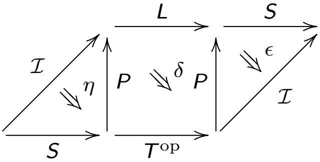

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Preservation of one-step expressiveness

Theorem

1. *The composition of two one-step expressive semantics remains one-step expressive.*
2. *A colimit of one-step expressive semantics remains expressive.*

Proof sketch.

1. By the fact that $(\delta_1 \otimes \delta_2)^* = \delta_1^* L_2 \circ T_1 \delta_2^*$ and T_i preserves $\mathcal{M}$-morphisms.
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Corollary (Labelling by A)

An A-fold coproduct of a one-step semantics (L, T, δ) is expressive if and only if (L, T, δ) is one-step expressive.

E.g.

1. Multi-modal logic for labelled image-finite Kripke frames (or descriptive general frames) is one-step expressive;
2. Probabilistic multi-modal logic is one-step expressive for labelled Markov chains.
3. Stochastic multi-modal logic is one-step expressive for labelled Markov processes.
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Conclusion

We have seen the following points in **CoLog**:

Modularity	Colimits, limits, and compositions.
Full semantics	Every logic for T-coalgebras can be translated to the full semantics.
Modality	Characterising modalities by Yoneda Lemma.
Generality	None of results depends on any particular propositional logic.

Thank you for your attention! Questions?

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