Derivations for appendix of "Light Scattering from Hair Fibers"

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1 Loci of reflections

Consider light arriving from a single direction, denoted by the unit vector ω_i. Let u be the axis of the hair (the tangent to the fiber). Wherever a ray traveling in the direction ω_i hits the cylinder, it will encounter a surface normal n that is perpendicular to u and that has $n \cdot u > 0$.

Since all the surface normals lie in the plane perpendicular to u (the "normal plane"), it's obvious from the symmetry of specular reflection that the directions in which the reflected rays leave the hair will all make the same angle with that plane as the incident direction does. To make this a little more formal, Figure 1 contains a diagram showing all these direction vectors drawn on the unit sphere.

In this figure, all the surface normals lie in the horizontal plane. Two particular surface normals n_1 and n_2 are drawn, together with the corresponding reflection vectors $\omega_{r,1}$ and $\omega_{r,2}$. The rule for specular reflection is that ω_i, n, and ω_r are coplanar and the components of ω_i and ω_r perpendicular to the normal are equal. In the diagram these projections are denoted h_i and h_r. For each reflection, h_i and h_r are equal in length and parallel, so the heights on the Gauss sphere of ω_i and ω_r above the normal plane are equal. This means that we know exactly where on the Gauss sphere we will find the surface reflection: on the circle parallel to the normal plane containing ω_i.

The rule for specular transmission (Snell's law) is similar: ω_i, n, and ω_t are coplanar and the components of ω_i and ω_t perpendicular to the normal have the constant ratio η. Figure 2 is analogous to the previous figure, but shows the refracted directions instead. Since the distances h_i and h_t are in the ratio η and are parallel, the heights on the Gauss sphere of ω_i and ω_t above the normal plane are also in the ratio η. This means the transmitted vectors all lie on a circle that is a factor of η closer to the normal plane than the incident vector, so they all make the same angle with the cylinder axis.

By the same argument, the rays that refract again on the way out of the cylinder will still all make the same angle with the axis. It's obvious from looking at the ray that passes through the axis that the angle of the twice-refracted rays is the same as the rays that are reflected off the surface. What's more, the rays can reflect inside the cylinder as many times as they want without changing

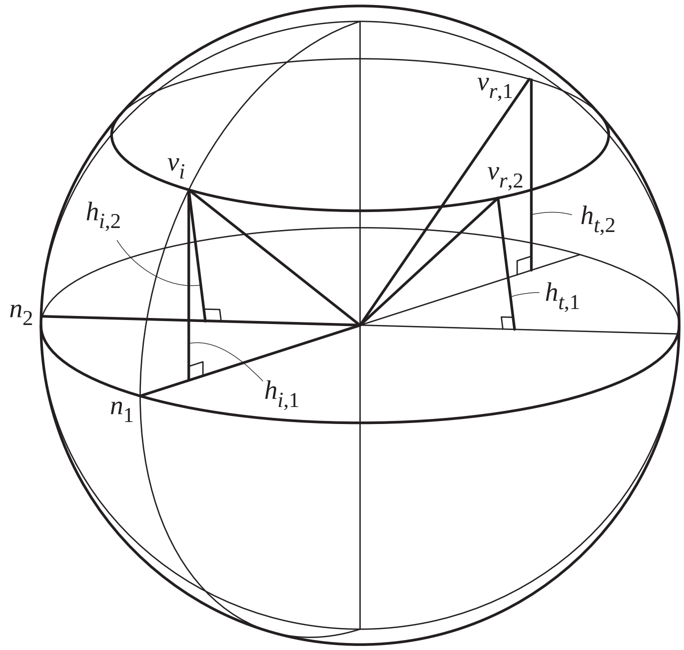

Figure 1: Reflected vectors from a cylinder lie in a cone.

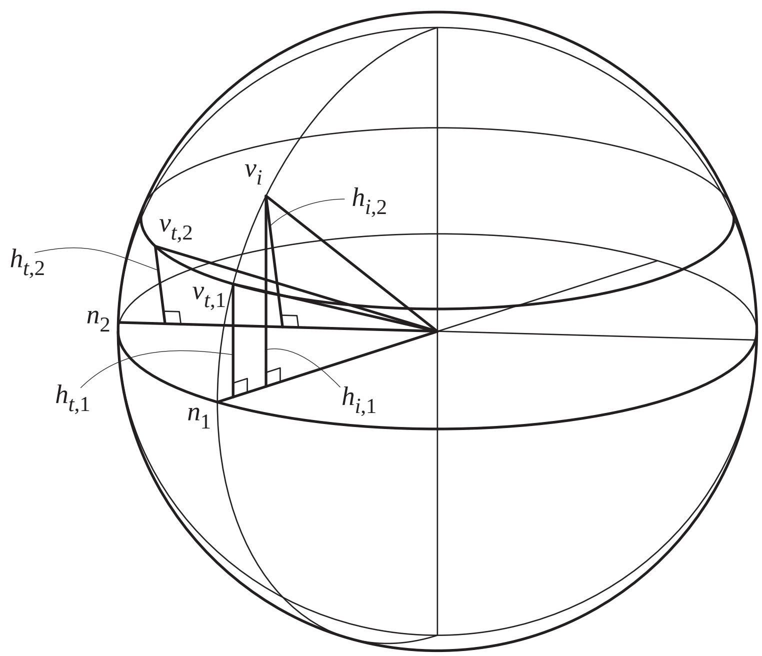

Figure 2: Refracted vectors from a cylinder lie in a cone.

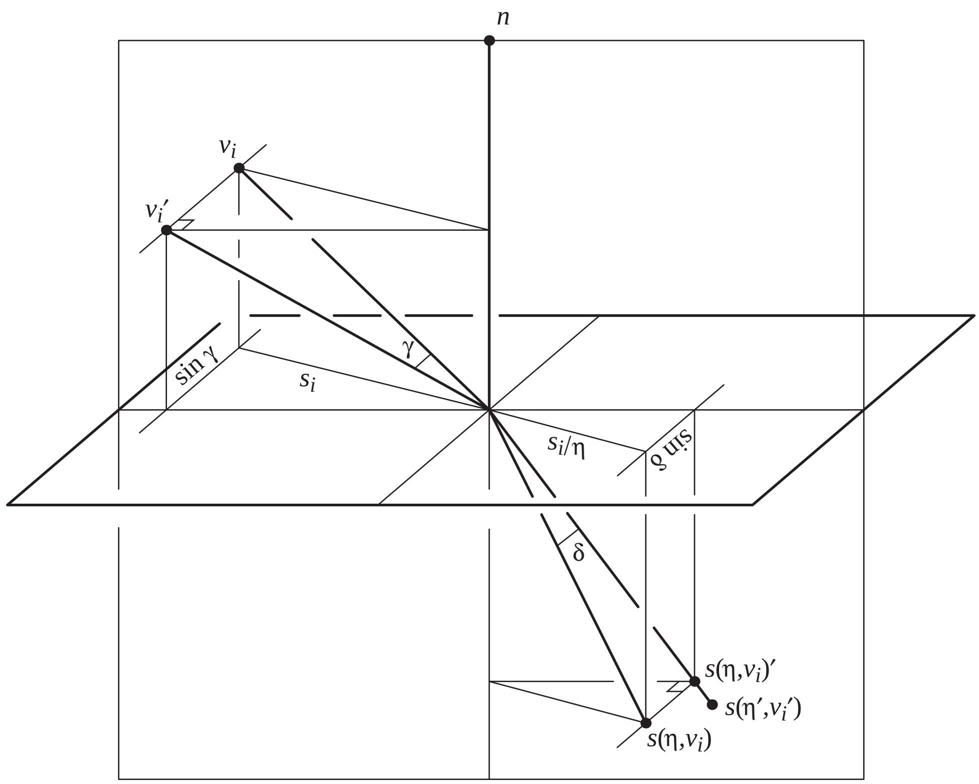

Figure 3: Derivation of the Bravais refractive index.

their inclination. This means we will find all rays that exit the cylinder, after any sequence of reflections and transmissions, in the same cone where we find the reflected rays! This holds for any cross section of the cylinder.

2 The Bravais index

When computing refractions, it is sometimes convenient to work with the projections of the direction vectors in question onto some convenient plane (in our case it will be a plane perpendicular to the hair's tangent). As illustrated in Figure 3, one cannot simply apply Snell's law to the projected vectors. In this figure, v_i is a direction vector coming into a horizontal surface with surface normal n, and $s(\eta, v_i)$ is the refracted vector, which is computed by applying Snell's law in the plane containing v_i and n. The key thing is that the projections of the incoming and refracted vectors into the surface plane are in the ratio η.

Looking at the projection into the plane of the paper, we have another similar diagram that has incoming direction v_i' and refracted direction $s(\eta, v_i)'$ with their projections in the proper ratio. The only thing keeping this from being another Snell's law diagram is that the lengths of these two vectors are not equal: $|v_i'| = \cos\gamma$ but $|s(\eta, v_i)'| = \cos\delta$. Scaling the refracted vector by $cos\gamma/\cos\delta$ produces a Snell's law diagram for refractive index $\eta' = \eta\cos\delta/\cos\gamma$. From the two similar triangles in the ground plane, $\eta\sin\delta = \sin\gamma$, from which it follows that $\eta^2\cos^2\delta = \eta^2 - sin^2\gamma$ and finally,

$$\eta' = \sqrt{\eta^2 - \sin^2\gamma} \Big/ \cos\gamma.$$

Incidentally, the fact that δ can be computed from just γ and η is another proof that the refracted directions form a cone about the tangent.

The implication of this is that for an incident ray that makes an angle γ with the projection plane, we can compute the projection of the refracted direction from the projection of the incident direction using the usual Snell's law but substituting the fictitious index of refraction η'. Since η' depends on γ, this is really only interesting if we have a whole set of rays that all have the same inclination. From the previous discussion, though, this is exactly what we have: the incident rays across the whole cylinder surface all have the same inclination to the normal plane, and since we showed that the reflected and refracted directions form cones centered on the hair tangent, we can continue to use this property to process further reflections and refractions.

3 Bravais and Fresnel

If we're going to compute reflected intensity we're going to need Fresnel's formulas too. Unfortunately, evaluating the Fresnel reflectance using the projected vectors and the Bravais refractive index does not work. However, it does work for the perpendicular-polarization component of the Fresnel reflectance; for the parallel-polarization component, we have to instead use the "anti-Bravais" refractive index:

$$\eta'' = \eta\cos\gamma/\cos\delta = \eta\cos\gamma \Big/ \sqrt{1 - \eta^{-2}\sin^2\gamma}\ .$$

Proof: First, the angles of the projected vectors with the normal satisfy $\cos\theta_i' = \cos\theta_i/\cos\gamma$ and $\cos\theta_t' = \cos\theta_t/\cos\delta$.

$$\begin{aligned} F_p &= \frac{\eta\cos\theta_i - \cos\theta_t}{\eta\cos\theta_i + \cos\theta_t} \\ &= \frac{\eta\frac{\cos\gamma}{\cos\delta}\frac{\cos\theta_i}{\cos\gamma} - \frac{\cos\theta_t}{\cos\delta}}{\eta\frac{\cos\gamma}{\cos\delta}\frac{\cos\theta_i}{\cos\gamma} + \frac{\cos\theta_t}{\cos\delta}} \\ &= \frac{\eta''\cos\theta_i' - \cos\theta_t'}{\eta''\cos\theta_i' + \cos\theta_t'} \end{aligned}$$

Similarly,

$$\begin{aligned} F_s &= \frac{\cos\theta_i - \eta\cos\theta_t}{\cos\theta_i + \eta\cos\theta_t} \\ &= \frac{\frac{\cos\theta_i}{\cos\gamma} - \eta\frac{\cos\delta}{\cos\gamma}\frac{\cos\theta_t}{\cos\delta}}{\frac{\cos\theta_i}{\cos\gamma} + \eta\frac{\cos\gamma}{\cos\gamma}\frac{\cos\theta_t}{\cos\delta}} \\ &= \frac{\cos\theta_i' - \eta'\cos\theta_t'}{\cos\theta_i' + \eta'\cos\theta_t'} \end{aligned}$$

Another way to write all this, which could be more symmetrical, is that Bravais gives you a "Bravais factor" $b(\eta,\gamma) = (\eta^2 - \sin^2\gamma)^{\frac{1}{2}}/\eta\cos\gamma$ and you use $b\eta$ for Snell and for perpendicular Fresnel and η/b for parallel Fresnel.