Efficient Minimization of New Quadric Metric for Simplifying Meshes with Appearance Attributes

(Addendum to IEEE Visualization 1999 paper)

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June 2000

Technical Report
MSR-TR-2000-64

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Abstract

In an earlier paper we introduced a new quadric metric for simplifying triangle meshes using the edge collapse operation. The quadric measures both the geometric accuracy of the simplified mesh surface and the fidelity of appearance fields defined on the surface (such as normals or colors). The minimization of this quadric metric involves solving a linear system of size $(3+m)\times(3+m)$, where m is the number of distinct appearance attributes. The system has only $O(m)$ nonzero entries, so it can be solved in $O(m^2)$ time using traditional sparse solvers such as the method of conjugate gradients. In this short addendum, we show that the special structure of the sparsity permits the system to be solved in $O(m)$ time.

1 Introduction

Complex triangle meshes arise naturally in many areas of computer graphics and visualization. In an earlier paper [2], we introduced a new quadric metric for simplifying such meshes using the edge collapse operation (Figure 1). The quadric measures both the geometric accuracy of the simplified mesh surface and the fidelity of appearance attributes defined on the surface (e.g. normals, colors).

Let m be the number of appearance attributes defined at the mesh vertices. Thus $m = 3$ for a mesh with (r, g, b) colors, and $m = 6$ for a mesh with both colors and normals. The particular quadric introduced in [2] has the nice property of consisting of a sparse matrix with only $O(m)$ nonzero entries.

For a given edge collapse, the optimal position and attributes of the resulting vertex are those that minimize the quadric functional. This minimization requires solving a linear system formed by the sparse matrix.

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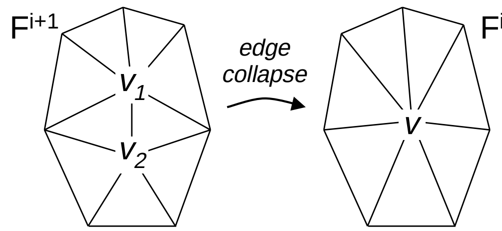

Figure 1: Edge collapse transformation.

In this short addendum, we show that the special structure of the sparse matrix allows the minimization to be solved in only $O(m)$ time.

2 Quadric metric definition

Each mesh vertex has a geometric position $\mathbf{p} \in \mathbf{R}^3$ and a set of m scalar attributes denoted by $\mathbf{s} \in \mathbf{R}^m$. These two elements form a column vector $\mathbf{v} = \left(\begin{smallmatrix} \mathbf{p} \\ \mathbf{s} \end{smallmatrix} \right) \in \mathbf{R}^{3+m}$.

For a given edge collapse, the quadric error $Q^v(\mathbf{v})$ at the resulting vertex v is defined as the sum of the quadrics defined by the faces F^{i+1} prior to the edge collapse. That is,

$$Q^v(\mathbf{v}) = \sum_{f \in F^{i+1}} \text{area}(f) \cdot Q^f(\mathbf{v}) \;, \tag{1}$$

where F^{i+1} are the faces adjacent to either v_1 or v_2 (see Figure 1), and the optional area(f) factors weigh the faces by their respective areas.

The quadric $Q^f(\mathbf{v})$ associated with each face is constructed based on the positions $(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3)$ and attributes $(\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3)$ at the three vertices of the face (see [2] for details). Let $\mathbf{n}$ be the face normal, $\mathbf{g}_1, \ldots, \mathbf{g}_m$ be the gradients of the attributes over the face, and the scalars $d, d_1, \ldots, d_m$ be defined by $d = \mathbf{n}^T\mathbf{p}_0$ and $d_j = \mathbf{g}_j^T\mathbf{p}_0 - \mathbf{s}_{0,j}$. (Note that each gradient $\mathbf{g}_j \in \mathbf{R}^3$ is a vector within the plane of the face, and that by construction, $d = \mathbf{n}^T\mathbf{p}_1 = \mathbf{n}^T\mathbf{p}_2$ and $d_j = \mathbf{g}_j^T\mathbf{p}_1 - \mathbf{s}_{1,j} = \mathbf{g}_j^T\mathbf{p}_2 - \mathbf{s}_{2,j}$.) Then,

$$Q^f(\mathbf{v}) = \mathbf{v}^T\mathbf{A}\mathbf{v} + 2\mathbf{b}^T\mathbf{v} + c \;,$$

with

$$\mathbf{A} = \left(\begin{array}{c|ccc} \mathbf{n}\mathbf{n}^T + \sum_j \mathbf{g}_j\mathbf{g}_j^T & -\mathbf{g}_1 & \cdots & -\mathbf{g}_m \\ \hline -\mathbf{g}_1^T & & & \\ \vdots & & I & \\ -\mathbf{g}_m^T & & & \end{array} \right), \quad \mathbf{b} = \left(\begin{array}{c} d\mathbf{n} + \sum_j d_j\mathbf{g}_j \\ \hline -d_1 \\ \vdots \\ -d_m \end{array} \right), \quad c = d^2 + \textstyle\sum_j d_j^2 \;.$$

Thus, $\mathbf{A}$ is a $(3+m)\times(3+m)$ symmetric matrix, $\mathbf{b}$ is a column vector of size $3+m$, and c is a scalar. The first 3 rows and first 3 columns of $\mathbf{A}$ are dense, but the remaining $m \times m$ submatrix is the identity matrix.

As a result, $Q^f(\mathbf{v})$ can be represented using exactly $10 + 4m$ coefficients, which is linear on the number m of appearance attributes.

To form the quadric Q^v at the vertex (see Equation 1), the corresponding entries of $(\mathbf{A}, \mathbf{b}, c)$ of the face quadrics Q^f are added together. Thus the matrix $\mathbf{A}$ for the vertex quadric Q^v has the form

$$\mathbf{A} = \begin{pmatrix} \mathbf{C} & \mathbf{B} \\ \mathbf{B}^T & \alpha I \end{pmatrix}$$

where $\mathbf{C}$ is a 3×3 matrix and I is still the $m \times m$ identity matrix. The additional degree of freedom $\alpha = \sum_{f \in F^{i+1}} \text{area}(f)$ brings the total number of coefficients to $11 + 4m$.

3 Quadric metric minimization

After an edge collapse, the vertex position $\mathbf{v}_{min}$ minimizing $Q^v(\mathbf{v})$ is found where the gradient ($\nabla Q^v(\mathbf{v}) = 2\mathbf{A}\mathbf{v} + 2\mathbf{b}$) equals zero, which is obtained by solving the linear system

$$\mathbf{A}\mathbf{v}_{min} = -\mathbf{b}\ .$$

The matrix $\mathbf{A}$ for a face quadric Q^f is rank-deficient; in fact, it has rank $m + 1$. The intuition is that $Q^f(\mathbf{v}) = 0$ if the position of $\mathbf{v}$ lies anywhere in the plane containing the face f and the attributes of $\mathbf{v}$ exactly extrapolate those of the face vertices. However, the matrix $\mathbf{A}$ for the vertex quadric Q^v, formed by summing the quadrics Q^f of its neighboring faces (Equation 1), has full rank if at least 3 of those neighboring faces have linearly independent normals.

Since the matrix $\mathbf{A}$ is sparse, the system can be solved iteratively using conjugate gradients in $O(m^2)$ time.

But in fact, the special structure of the sparse matrix allows the system to be solved in $O(m)$ time, as shown next. With the block partition of $\mathbf{A}$ introduced above, the system can be rewritten as

$$\begin{pmatrix} \mathbf{C} & \mathbf{B} \\ \mathbf{B}^T & \alpha I \end{pmatrix} \begin{pmatrix} \mathbf{p}_{min} \\ \mathbf{s}_{min} \end{pmatrix} = \begin{pmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \end{pmatrix}\ .$$

In the above system, pre-multiply the last set of m rows by $-\mathbf{B}/\alpha$ and add the result to the first set of 3 rows. The result is

$$(\mathbf{C} - \frac{1}{\alpha}\mathbf{B}\mathbf{B}^T)\mathbf{p}_{min} = \mathbf{b}_1 - \frac{1}{\alpha}\mathbf{B}\mathbf{b}_2$$

which is a symmetric 3×3 linear system. Solve this system to obtain the optimal vertex position $\mathbf{p}_{min}$. Then, back-substitute into the last set of m rows to obtain the optimal vertex attributes $\mathbf{s}_{min}$:

$$\mathbf{s}_{min} = \frac{1}{\alpha}(\mathbf{b}_2 - \mathbf{B}^T\mathbf{p}_{min})\ .$$

Solving the 3×3 linear system takes only constant time. Computing $\mathbf{B}\mathbf{B}^T$ and $\mathbf{B}\mathbf{b}_2$, and back-substituting to compute $\mathbf{s}_{min}$ both require $O(m)$ time.

4 Volume preservation

Our earlier paper [2] showed that simplification could be guaranteed to preserve the overall mesh volume by adding a linear constraint $\mathbf{g}_{VOL}^T\,\mathbf{p} + d_{VOL} = 0$ on the vertex position $\mathbf{p}$ for each edge collapse. This linear constraint is easily achieved using a Lagrange multiplier, which adds one more row and column to the system [1]. Our $O(m)$ solution scheme extends trivially to this case by growing $\mathbf{C}$ to be a 4×4 submatrix that includes this Lagrange multiplier constraint.

5 Implementation results

We have implemented the quadric metric minimization using an $O(m^3)$ general linear system solver, and our new $O(m)$ scheme. For the case of meshes with both normals and colors (i.e. $m = 6$), we find that our new scheme is about 2.5 times faster than the general solver on a 750 MHz Pentium III PC.

References

[1] GOLUB, G., AND VAN LOAN, C. *Matrix Computations*, 3rd ed. John Hopkins University Press, 1996.

[2] HOPPE, H. New quadric metric for simplifying meshes with appearance attributes. In *Visualization '99 Proceedings* (1999), IEEE, pp. 59–66.