On Random Sampling Auctions for Digital Goods

Saeed Alaei   Azarakhsh Malekian   Aravind Srinivasan

Outline

Truthful Auctions (Brief Review)

What is a truthful auction?

Any auction where disclosing the private information is a weakly dominant strategy for bidders (e.g. second price auction).

Truthful Auctions (Brief Review)

What is a truthful auction?

Any auction where disclosing the private information is a weakly dominant strategy for bidders (e.g. second price auction).

Why truthful auctions?

- Simpler to analyze (efficiency, revenue, . . .)
- Simpler for bidders (no strategic behavior)
- Smaller space of possible mechanisms for auction designer to look at

Truthful Auctions (Brief Review)

What is a truthful auction?

Any auction where disclosing the private information is a weakly dominant strategy for bidders (e.g. second price auction).

Why truthful auctions?

- Simpler to analyze (efficiency, revenue, . . .)
- Simpler for bidders (no strategic behavior)
- Smaller space of possible mechanisms for auction designer to look at

Theorem (Revelation Principle)

Any non-truthful mechanism that has a Nash Equilibrium can be converted to a truthful mechanism.

Vickrey-Clarke-Groves (VCG)

Abstract Model

- A set of outcomes $A = \{a_1, \cdots, a_m\}$.
- A set of bidders N, each having valuation $v_i(a)$ for each $a \in A$.
- The utility of bidder i is $u_i = v_i(a) - p_i$ where p_i is payment.
- The utility of the auctioneer is $u_0 = \sum_{i \in N} p_i$.
- The social welfare is $U_N(a) = \sum_{i \in N} u_i(a) \ = \sum_{i \in N} v_i(a)$.

Vickrey-Clarke-Groves (VCG)

Abstract Model

- A set of outcomes $A = \{a_1, \cdots, a_m\}$.
- A set of bidders N, each having valuation $v_i(a)$ for each $a \in A$.
- The utility of bidder i is $u_i = v_i(a) - p_i$ where p_i is payment.
- The utility of the auctioneer is $u_0 = \sum_{i \in N} p_i$.
- The social welfare is $U_N(a) = \sum_{i \in N} u_i(a) \; = \sum_{i \in N} v_i(a)$.

Definition (VCG Mechanism)

1. Ask bidders to submit their private valuations v_i.
2. Choose the outcome $a^* = \text{argmax}_a \, U_N(a)$.
3. Set the payment of each bidder i to
   $p_i = \left(\max_a U_{N \setminus \{i\}}(a)\right) - \left(U_N(a^*) - v_i(a^*)\right)$

Truthful Mechanisms

Abstract Model

- Each bidder has a multidimensional type vector $\mathbf{t}_i$
- Let $x_{\mathbf{t}_{-i}}(\mathbf{b}_i)$ be the allocation function of mechanism for i.
- Let $u_i(x_{\mathbf{t}_{-i}}(\mathbf{b}_i), \mathbf{t}_i)$ be the utility of advertiser i if she submits $\mathbf{b}_i$ while her true type is $\mathbf{t}_i$.

Theorem (Characterization of Truthful Mechanisms)

An allocation mechanism x is truthful if the following payment is well-defined:

$$p_i = \int_0^{\mathbf{t}_i} \nabla_{\mathbf{t}_i} u_i(x_{\mathbf{t}_{-i}}(\mathbf{b}_i), \mathbf{b}_i) \cdot d\mathbf{b}_i$$

Truthful Mechanisms for Single Parameter setting

Abstract Model

- Each bidder has a single parameter type v_i, the value for the item
- Let $x_{\mathbf{v}_{-i}}(b_i)$ be the allocation function of mechanism for i.
- Let $v_i x_{\mathbf{v}_{-i}}(b_i)$ be the utility of advertiser i if she submits b_i while her true type is v_i.

Theorem (Characterization of Truthful Mechanisms)

An allocation mechanism x is truthful if it is monotone (increasing) and its payment is:

$$p_i = \int_0^{v_i} b \, \frac{\partial}{\partial b} x_{\mathbf{v}_{-i}}(b) db = v_i x_{\mathbf{v}_{-i}}(v_i) - \int_0^{v_i} x_{\mathbf{v}_{-i}}(b) db$$

Myerson Optimal Auction

Model

We have a single item to sell. Bidders have unit demand and pure private valuations and bidder i's type, v_i, is drawn independently from the distribution $F_i(v)$. We are looking for an auction that maximizes revenue in expectation.

Theorem (Optimal Bayesian Auction)

For each bidder, compute the virtual valuation, $\phi_i(v_i)$. Give the item to the bidder i with the highest positive virtual valuation and charge her equal to $\phi_i^{-1}(\phi_j(v_j))$ where $\phi_j(v_j)$ is the second highest virtual valuation or $\phi_i^{-1}(0)$ if all others are negative.

$$\phi_i(v_i) = v_i - \frac{1 - F_i(v_i)}{f_i(v_i)} \tag{1}$$

Digital Goods Auction, Problem Definition

- Originally proposed by Goldberg & Hartline.
- We have a single type of good with unlimited supply
- There are n bidders with bids $v_1 \geq \cdots \geq v_n$.
- We want a revenue-maximizing incentive compatible auction.
- We have no prior information on distributions.
- Benchmark is the optimal uniform price auction: $\max_{\lambda \geq 2} \; \lambda \cdot \mathbf{v}_\lambda$

Random Sampling Optimal Price Auction

- The mechanism:
  - Partition the bids to two groups A and B uniformly at random.
  - Compute the optimal unform price in each group and offer it to the other group.

Random Sampling Optimal Price Auction

- The mechanism:
  - Partition the bids to two groups A and B uniformly at random.
  - Compute the optimal unform price in each group and offer it to the other group.
- RSOP is incentive compatible.

Random Sampling Optimal Price Auction

- The mechanism:
  - Partition the bids to two groups A and B uniformly at random.
  - Compute the optimal unform price in each group and offer it to the other group.
- RSOP is incentive compatible.

Conjecture

The revenue of RSOP is at least $\frac{1}{4}OPT$. i.e. RSOP is 4-competitive.

RSOP Example

- Suppose the bids are $\{7, 6, 5, 1\}$.
- After random partitioning of the bids, $A = \{6, 1\}$ and $B = \{7, 5\}$.
- We offer 6 to B and 5 to A.
- we get a revenue of 11 while OPT is 15.

Conjecture

The worst case performance of RSOP is when bids are $\{1, \frac{1}{2}\}$.

Previous/Present Results

- Goldberg & Hartline (2001) : $\frac{OPT}{RSOP} < 7600$
- Feige et al (2005) : $\frac{OPT}{RSOP} < 15$
- Our result (2008) : $\frac{OPT}{RSOP} < 4.68$

Previous/Present Results

- Goldberg & Hartline (2001) : $\frac{OPT}{RSOP} < 7600$
- Feige et al (2005) : $\frac{OPT}{RSOP} < 15$
- Our result (2008) : $\frac{OPT}{RSOP} < 4.68$

Theorem

The competitive ratio of RSOP is (λ is the index of the winning bid in OPT) (e.g. in $\{7, 6, 5, 1\}$, $\lambda = 3$):

$$\begin{cases} < 4.68 & \lambda < 6 \\ < 4 & \lambda > 6 \\ < 3.3 & \lambda \to \infty \end{cases} \quad (2)$$

Assumptions

- We have an infinite number of bids (i.e. $n = \infty$), by adding 0's.

Assumptions

- We have an infinite number of bids (i.e. $n = \infty$), by adding 0's.
- $OPT = 1$, by scaling all the bids.

Assumptions

- We have an infinite number of bids (i.e. $n = \infty$), by adding 0's.
- $OPT = 1$, by scaling all the bids.
- v_1 is always in B and we only consider the revenue obtained from set B.

A lowerbound on RSOP revenue when $\lambda > 10$

- A dynamic programming method for computing the lower bound given the λ.

A lowerbound on RSOP revenue when $\lambda > 10$

- A dynamic programming method for computing the lower bound given the λ.
- A second method which is independent of λ but assumes it is large (i.e. > 5000) and uses Chernoff bound.

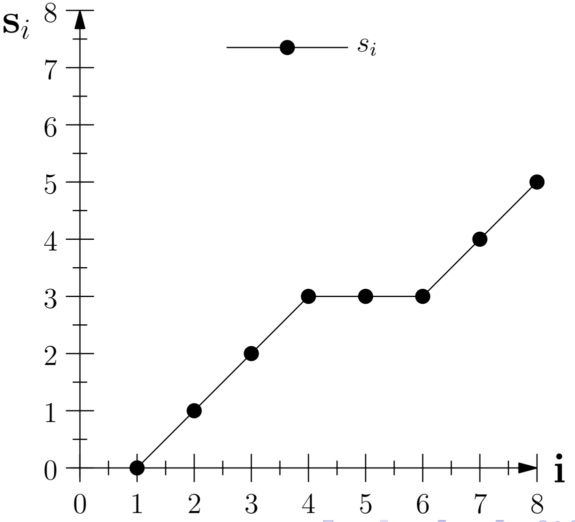
Random Partition

Example

$$A = \{v_2, v_3, v_4, v_7, v_8\}$$
$$B = \{v_1, v_5, v_6\}$$

Definition

$$S_i = \#\{v_j | v_j \in A, j \leq i\}$$



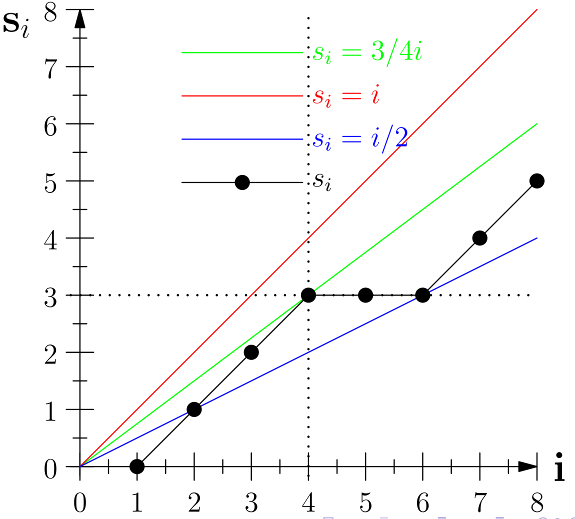
Random Partition

Example

$$A = \{v_2, v_3, v_4, v_7, v_8\}$$
$$B = \{v_1, v_5, v_6\}$$

Definition

$$S_i = \#\{v_j | v_j \in A, j \leq i\}$$



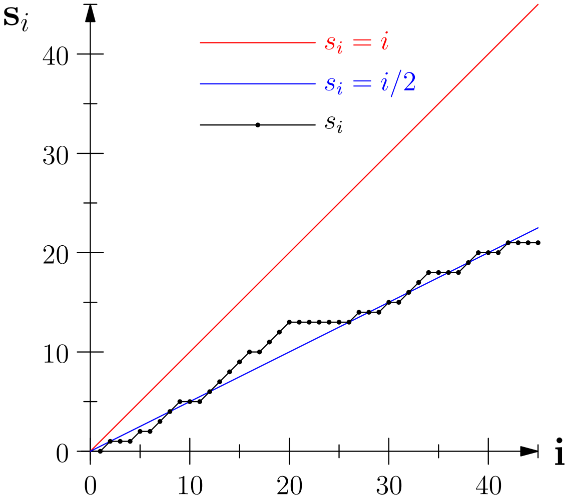
Random Partition

Observation

$$\lim_{i\to\infty} \frac{S_i}{i} \to \frac{1}{2}$$

or

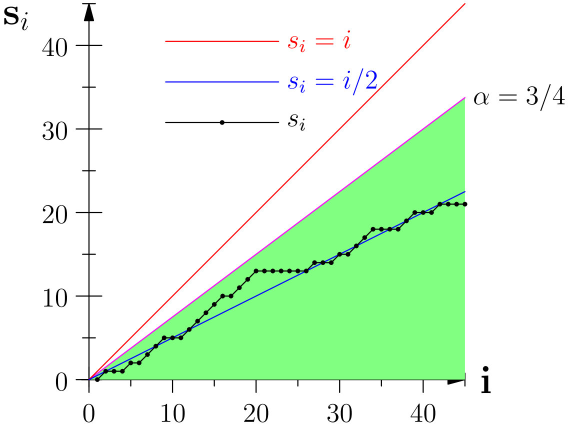
$$\lim_{i\to\infty} Pr\left[\frac{S_i}{i} < \frac{1}{2} - \epsilon\right] \to 0$$



Worst Profit Ratio

Observations

$$\forall j : \quad \frac{S_j}{j} < \alpha$$

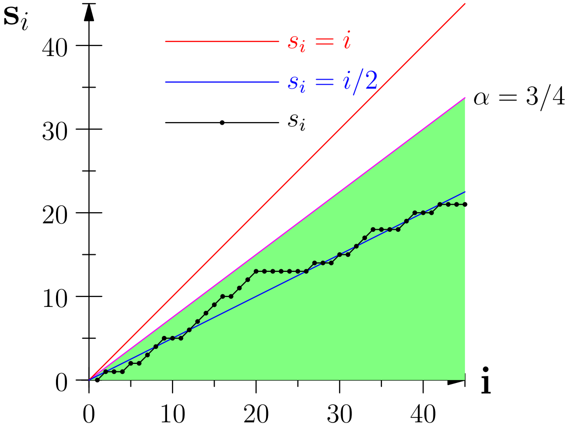


Worst Profit Ratio

Observations

$$\forall j : \quad \frac{S_j}{j} < \alpha$$

$$\frac{Prof(B)}{Prof(A)} \geq \frac{1-\alpha}{\alpha}$$



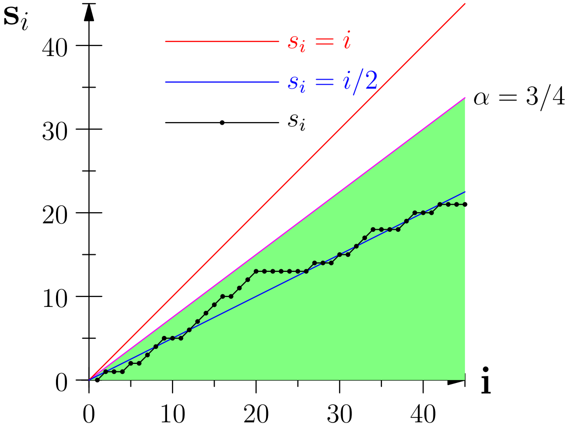
Worst Profit Ratio

Observations

$$\forall j: \quad \frac{S_j}{j} < \alpha$$

$$\frac{Prof(B)}{Prof(A)} \geq \frac{1-\alpha}{\alpha}$$

$$Prof(A) \geq \frac{S_\lambda}{\lambda}$$



Worst Profit Ratio

Observations

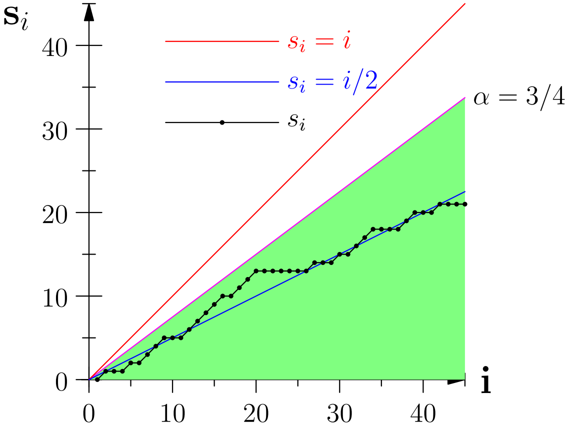
$$\forall j: \quad \frac{S_j}{j} < \alpha$$

$$\frac{Prof(B)}{Prof(A)} \geq \frac{1-\alpha}{\alpha}$$

$$Prof(A) \geq \frac{S_\lambda}{\lambda}$$

$$Z = \min_i \frac{i - S_i}{S_i}$$

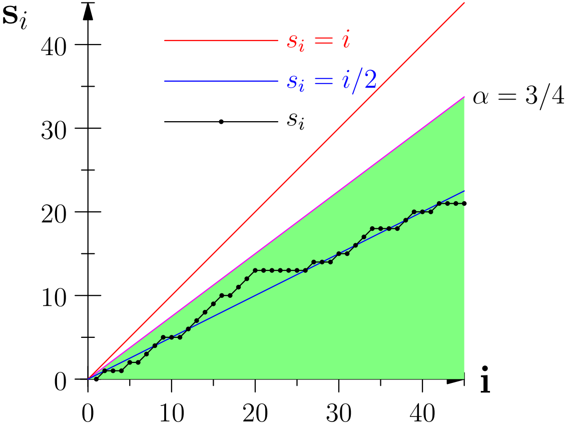
$$Prof(B) \geq E[Z\frac{S_\lambda}{\lambda}]$$



α-Event

Definition ($\mathcal{E}_\alpha$ event)

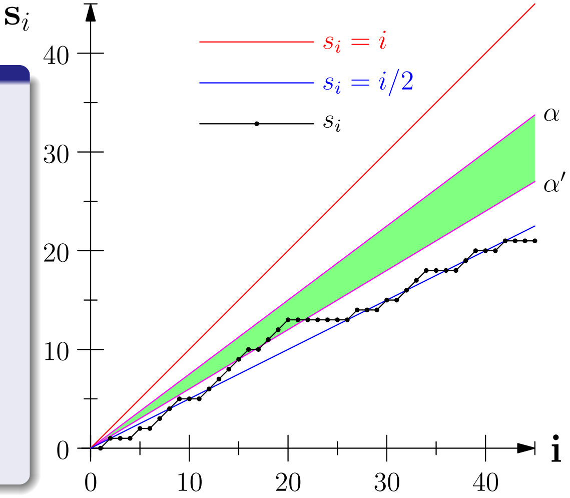
$$\mathcal{E}_\alpha : \quad \forall j : \frac{S_j}{j} \leq \alpha$$



α-Events

$$\mathcal{E}_{[\alpha',\alpha]} = \mathcal{E}_\alpha - \mathcal{E}_{\alpha'}$$

$$Z|\mathcal{E}_{[\alpha',\alpha]} \geq \frac{1-\alpha}{\alpha}$$



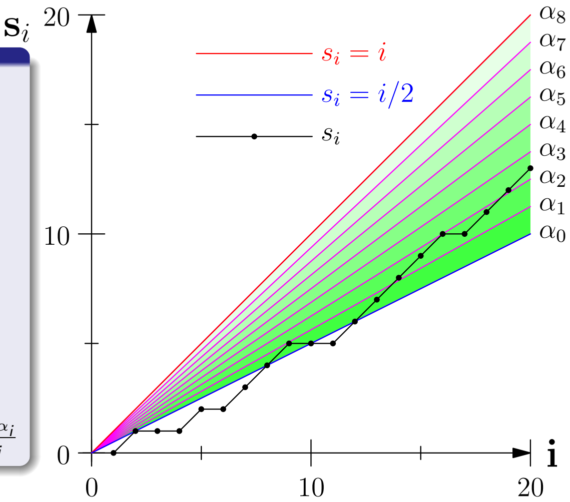
α-Events

$$\mathcal{E}_{[\alpha',\alpha]} = \mathcal{E}_{\alpha} - \mathcal{E}_{\alpha'}$$

$$Z|\mathcal{E}_{[\alpha',\alpha]} \geq \frac{1-\alpha}{\alpha}$$

$$E[Z] \geq \sum_i Pr[\mathcal{E}_{[\alpha_i,\alpha_{i+1}]}] \frac{1-\alpha_i}{\alpha_i}$$

$$E[Z] \geq \sum_i (Pr[\mathcal{E}_{\alpha_{i+1}}] - Pr[\mathcal{E}_{\alpha_i}]) \frac{1-\alpha_i}{\alpha_i}$$



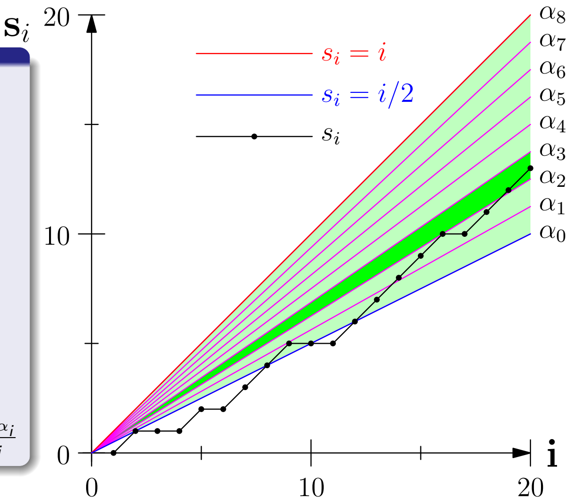
α-Events

$$\mathcal{E}_{[\alpha',\alpha]} = \mathcal{E}_{\alpha} - \mathcal{E}_{\alpha'}$$

$$Z|\mathcal{E}_{[\alpha',\alpha]} \geq \frac{1-\alpha}{\alpha}$$

$$E[Z] \geq \sum_i Pr[\mathcal{E}_{[\alpha_i,\alpha_{i+1}]}]\frac{1-\alpha_i}{\alpha_i}$$

$$E[Z] \geq \sum_i (Pr[\mathcal{E}_{\alpha_{i+1}}] - Pr[\mathcal{E}_{\alpha_i}])\frac{1-\alpha_i}{\alpha_i}$$



Computing $E[Z]$

Lemma

The worst ratio of profit of set B to profit of set A can be computed using the following:

$$E[Z] = \sum_i Pr[\mathcal{E}_{[\alpha_{i-1},\alpha_i]}] \frac{1-\alpha_i}{\alpha_i}$$

$$= \sum_i (Pr[\mathcal{E}_{\alpha_i}] - Pr[\mathcal{E}_{\alpha_{i-1}}]) \frac{1-\alpha_i}{\alpha_i}$$

The Dynamic Program for computing $P[\mathcal{E}_\alpha]$

Definition

Let $P_\alpha(k,j)$ be the probability that for any $1 \leq i \leq k$, at most α fraction of the $v_1, \ldots, v_i$ are in A and exactly j of $v_1, \cdots, v_k$ are in A. Let $P_\alpha(k) = \sum_{j=0}^{k} P_\alpha(k,j)$, then $Pr[\mathcal{E}_\alpha] = P_\alpha(\infty)$

Dynamic Program for computing $P_\alpha(k,j)$

$$P_\alpha(k,j) = \begin{cases} 0 & j > \alpha k \\ 1 & j = k = 0 \\ 1/2 P_\alpha(k-1,j) & j = 0, k > 0 \\ 1/2 P_\alpha(k-1,j) + 1/2 P_\alpha(k-1,j-1) & 0 < j < \alpha k \end{cases}$$

When λ is large

Claim

As λ increases, the correlation between S_λ/λ and Z decreases so we can separate them.

$$\begin{aligned} Prof(b) &\geq E\left[\frac{S_\lambda}{\lambda} Z\right] \\ &\approx E[\frac{S_\lambda}{\lambda}]E[Z] \\ &\approx \frac{1}{2}E[Z] \end{aligned}$$

We use a variant of Chernoff bound to bound the error caused by separating the two terms.

The Dynamic Program for $E[\frac{S_\lambda}{\lambda} Z]$

Definition

Let $R_\alpha(k, j)$ the expected value of lowerbound for profit of set A conditioned and multiplied by the probability that for any $1 \leq i \leq k$, at most α fraction of the $v_1, \ldots, v_i$ are in A and exactly j of $v_1, \cdots, v_k$ are in A.

Dynamic Program for computing $R_\alpha(k, j)$

$$R_\alpha(k,j) = \begin{cases} 0 & \begin{array}{l} j = 0 \text{ or} \\ j > \alpha k \end{array} \\ 1/2 R_\alpha(k-1, j) + 1/2 R_\alpha(k-1, j-1) & 0 < j \leq \alpha k \\ \frac{j}{\lambda} P_\alpha(k-1, j) & k = \lambda \end{cases}$$

The Dynamic Program for $E[\frac{S_\lambda}{\lambda} Z]$ (Continued)

Dynamic Program for computing $E\left[\frac{S_\lambda}{\lambda} Z \middle| \mathcal{E}_\alpha\right]$

$$R_\alpha(k) = \sum_{i=0}^{j} R_\alpha(k, j)$$

$$R_\alpha(\infty) = E\left[\frac{S_\lambda}{\lambda} \middle| \mathcal{E}_\alpha\right] Pr\left[\mathcal{E}_\alpha\right]$$

$$E[\frac{S_\lambda}{\lambda} Z] = \sum_i (R_{\alpha_i} - R_{\alpha_{i-1}}) \frac{1 - \alpha_i}{\alpha_i}$$

An upperbound on the revenue of RSOP with large λ

Theorem

For any given λ, there is a set of bids with λ being the index of the winning price and such that RSOP does not get a revenue of more than $3/8$.

The equal revenue instances

Definition

An Equal Revenue Instance with n bids consists of the bids $\{1, \frac{1}{2}, \cdots, \frac{1}{n}\}$.

The equal revenue instances

Definition

An Equal Revenue Instance with n bids consists of the bids $\{1, \frac{1}{2}, \cdots, \frac{1}{n}\}$.

Observation

In an equal revenue instance, the price offered from each set is the worst price for the other set.

The equal revenue instances, RSOP'

Definition ($RSOP'$)

It is the same as RSOP except that when set A is empty, the price that is offered from A to B is v_n instead of 0. The difference between the revenue of RSOP and RSOP' is $1/2^n$.

The equal revenue instances, RSOP'

Definition ($RSOP'$)

It is the same as RSOP except that when set A is empty, the price that is offered from A to B is v_n instead of 0. The difference between the revenue of RSOP and RSOP' is $1/2^n$.

Claim

The revenue of RSOP' on an equal revenue instance with $n+1$ bids is less than that with n bids. The proof is by induction.

The equal revenue instances, RSOP'

Definition ($RSOP'$)

It is the same as RSOP except that when set A is empty, the price that is offered from A to B is v_n instead of 0. The difference between the revenue of RSOP and RSOP' is $1/2^n$.

Claim

The revenue of RSOP' on an equal revenue instance with $n+1$ bids is less than that with n bids. The proof is by induction.

Fact

Revenue of RSOP for equal revenue instances with $n \leq 10$ is at most $\frac{1}{2.65}$.

Revenue

RSOP revenue (basic lowerbound)

λ	$E[RSOP]$	Competitive-Ratio
2	0.125148	7.99
3	0.166930	5.99
4	0.192439	5.20
5	0.209222	4.78
6	0.221407	4.52
7	0.230605	4.34
8	0.237862	4.20
9	0.243764	4.10
10	0.248647	4.02
15	0.264398	3.78
20	0.273005	3.66
100	0.296993	3.37
500	0.302792	3.30
1000	0.303560	3.29
1500	0.303818	3.29
2000	0.303949	3.29

Based on dynamic programming up to $n = 5000$ and then Chernoff bound.

RSOP revenue (secondary lowerbound)

λ	$E[RSOP]$	Competitive-Ratio
2	0.2138	4.68
3	0.2178	4.59
4	0.238	4.20
5	0.243	4.11
6	0.2503	3.99
7	0.2545	3.93
8	0.2602	3.84
9	0.2627	3.81
10	0.2669	3.75

Questions?

Questions ?