Tensor Factorization via Matrix Factorization

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Arun Tejasvi Chaganty*
Percy Liang

Stanford University

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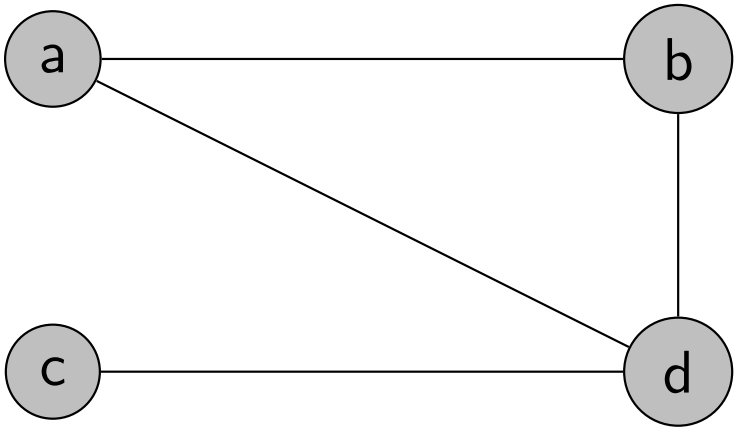
An application: community detection



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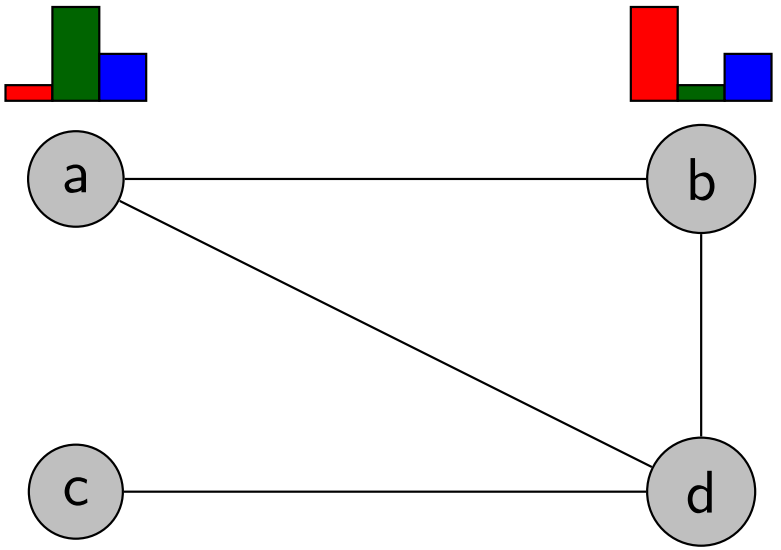

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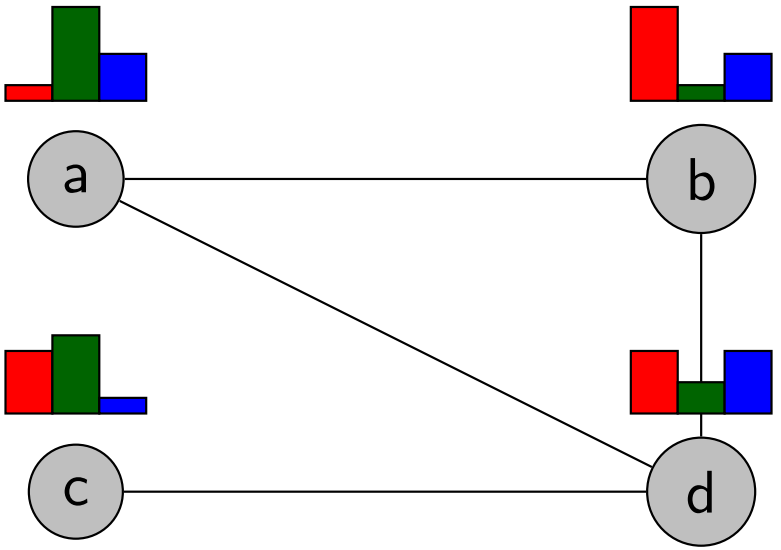

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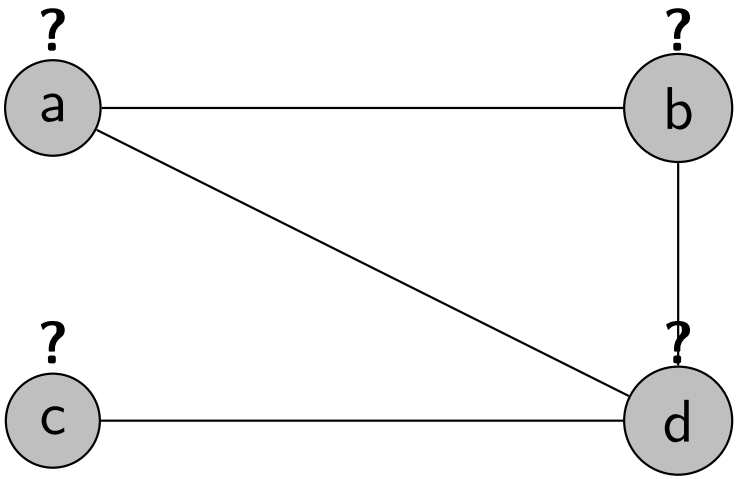
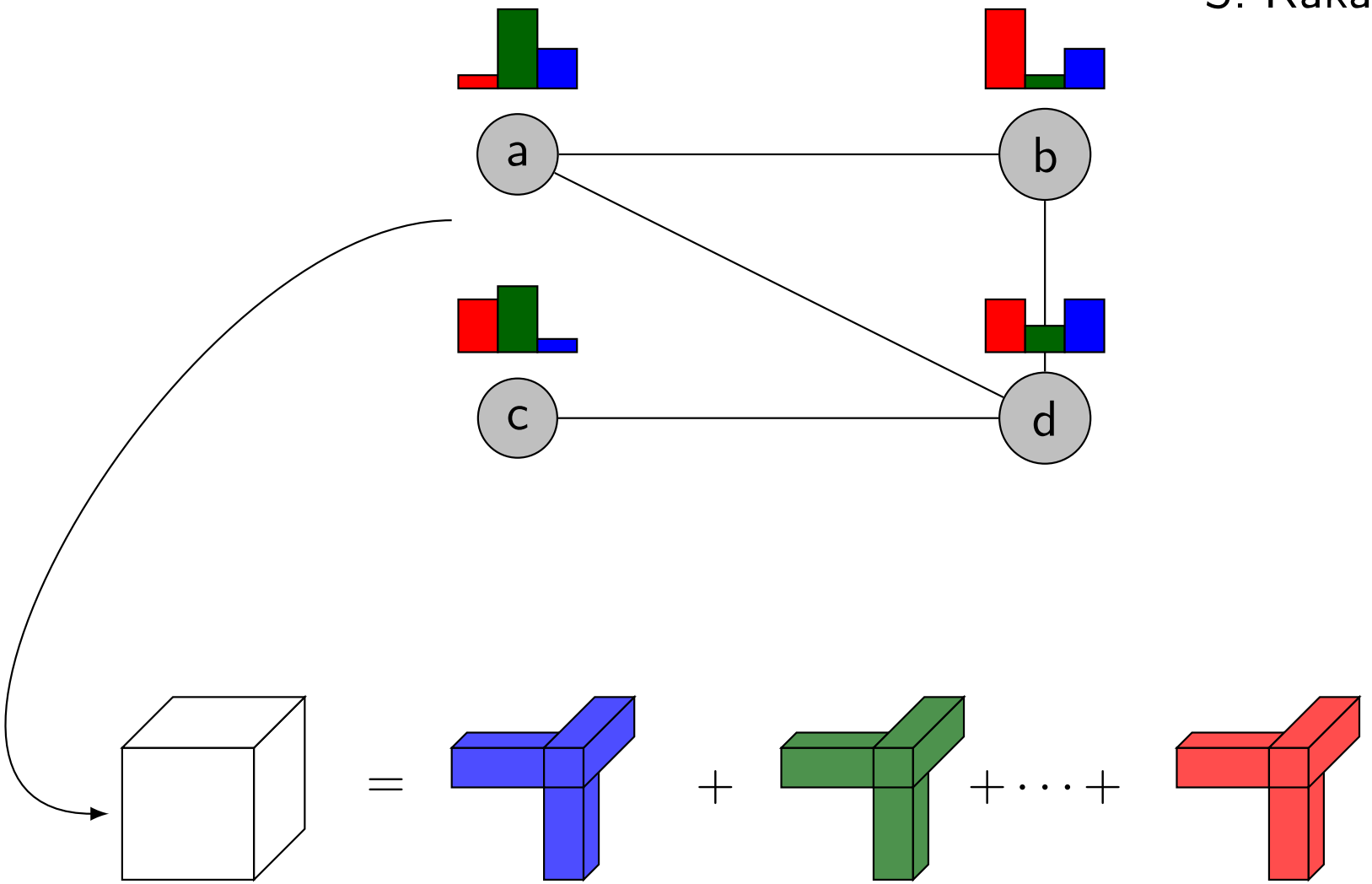


An application: community detection

Anandkumar, Ge, Hsu, and S. Kakade 2013



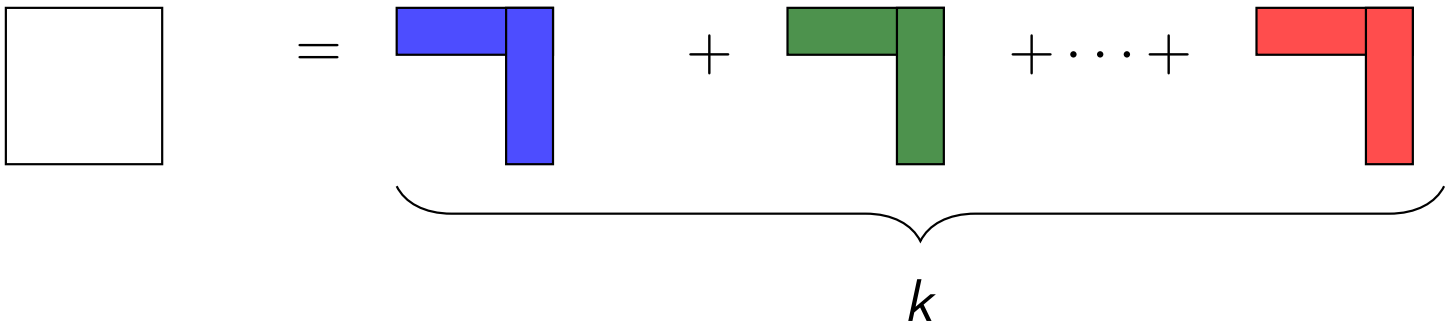
Applications of tensor factorization

- **Community detection**
  - Anandkumar, Ge, Hsu, and S. Kakade 2013
- **Parsing**
  - Cohen, Satta, and Collins 2013
- **Knowledge base completion**
  - Chang et al. 2014
  - Singh, Rocktäschel, and Riedel 2015
- **Topic modelling**
  - Anandkumar, Foster, et al. 2012
- **Crowdsourcing**
  - Zhang et al. 2014
- **Mixture models**
  - Anandkumar, Ge, Hsu, S. M. Kakade, et al. 2013
- **Bottlenecked models**
  - Chaganty and Liang 2014
- ...

What is tensor (CP) factorization?

- Tensor analogue of matrix eigen-decomposition.

$$M = \sum_{i=1}^{k} \pi_i u_i \otimes u_i \qquad .$$



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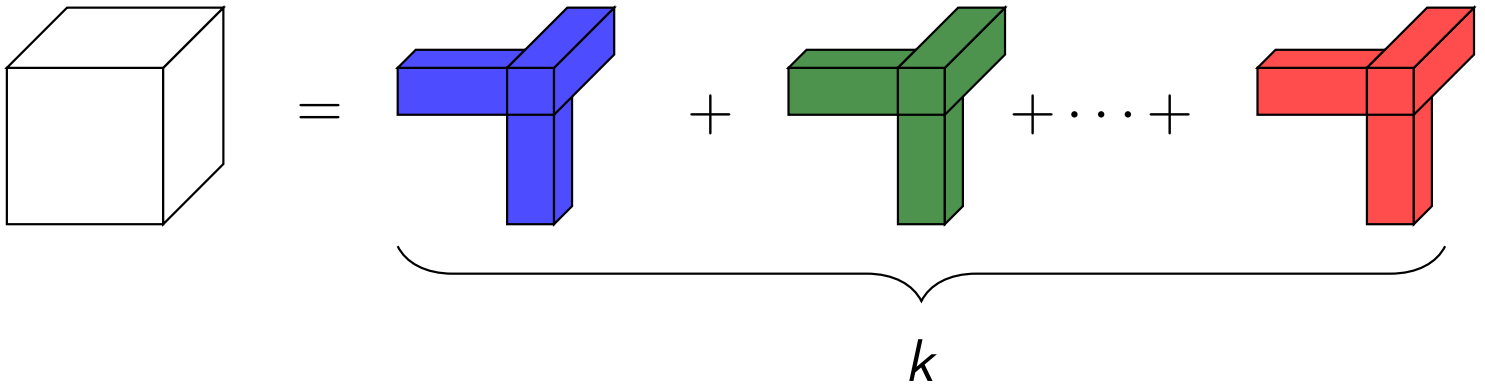
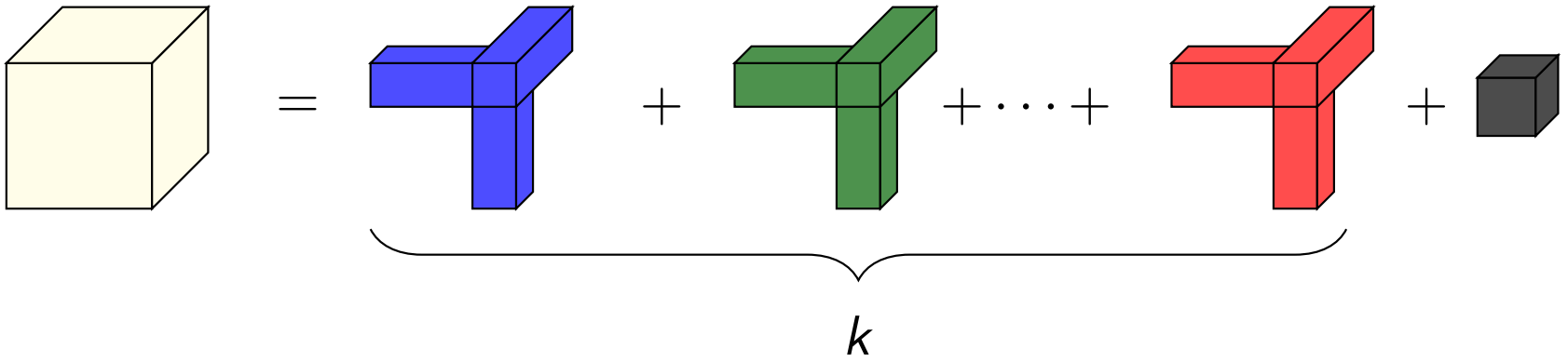


What is tensor (CP) factorization?

- Tensor analogue of matrix eigen-decomposition.

$$\widehat{T} = \sum_{i=1}^{k} \pi_i u_i \otimes u_i \otimes u_i + \epsilon R.$$

- **Goal:** Given T with noise, ϵR, **recover factors** u_i.



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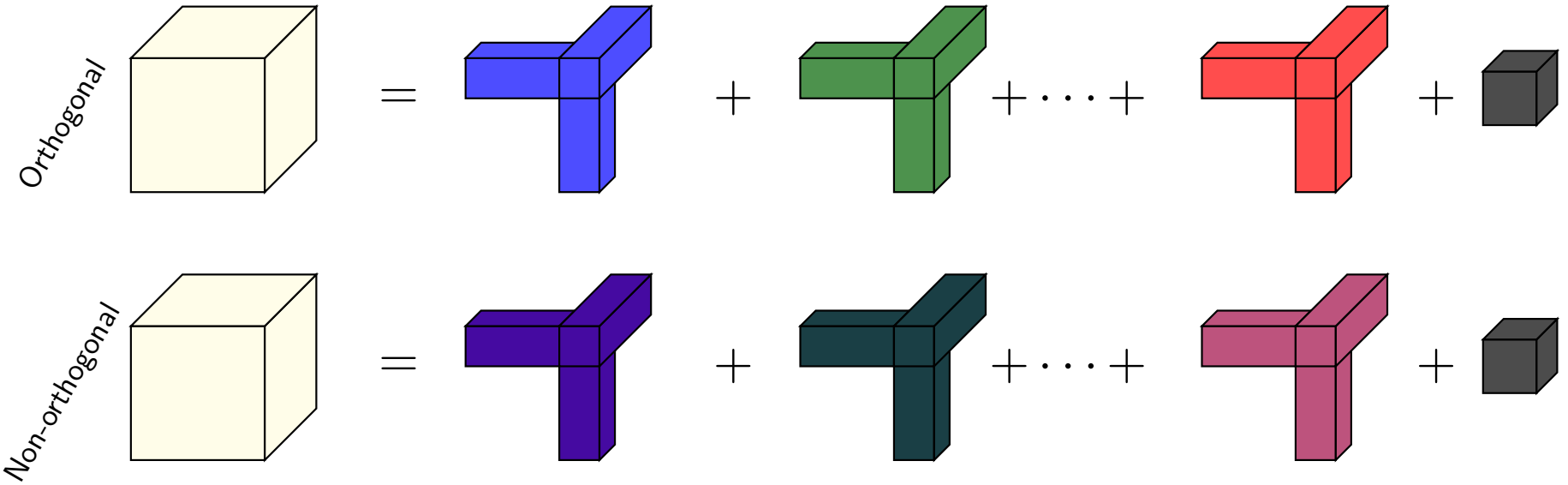

Existing tensor factorization algorithms

- **Tensor power method** (Anandkumar, Ge, Hsu, S. M. Kakade, et al. 2013)
  - Analog of matrix power method.
  - Sensitive to noise.
  - Restricted to orthogonal tensors.

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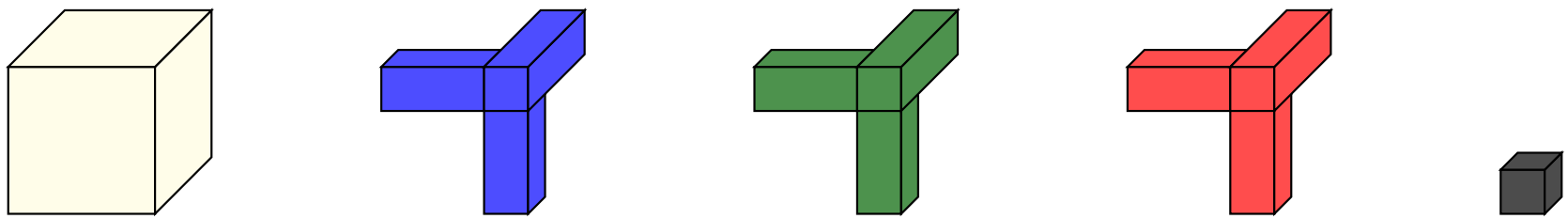
Our approach: reduce to existing **fast and robust matrix algorithms**.

Outline

Tensor factorization via single matrix factorization



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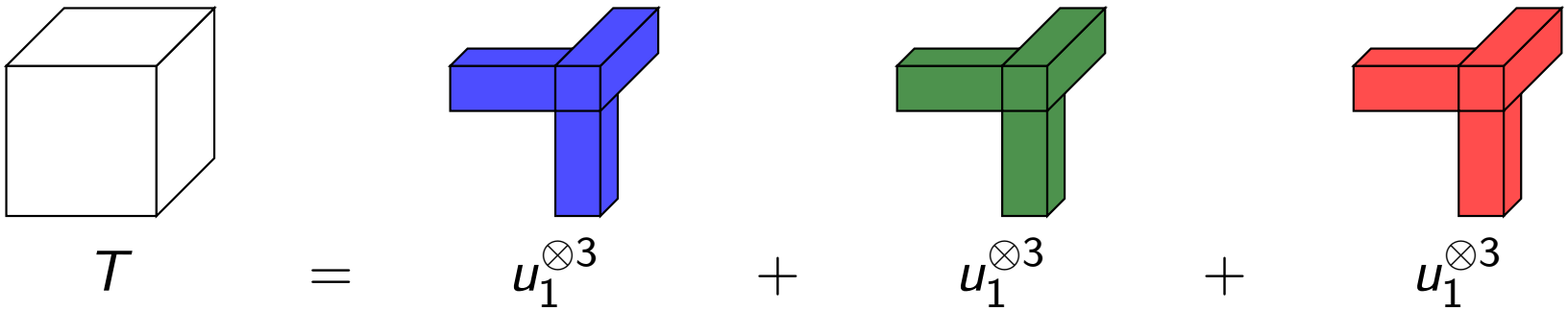

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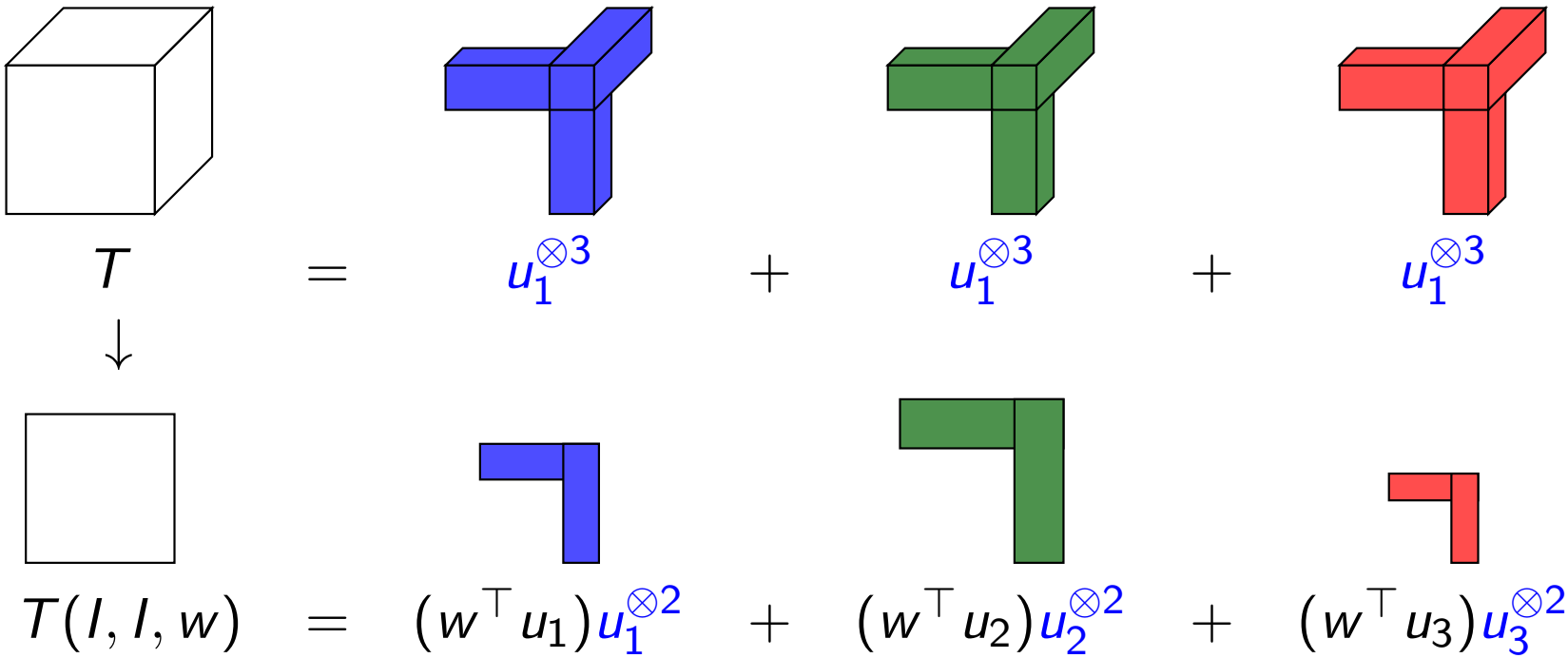

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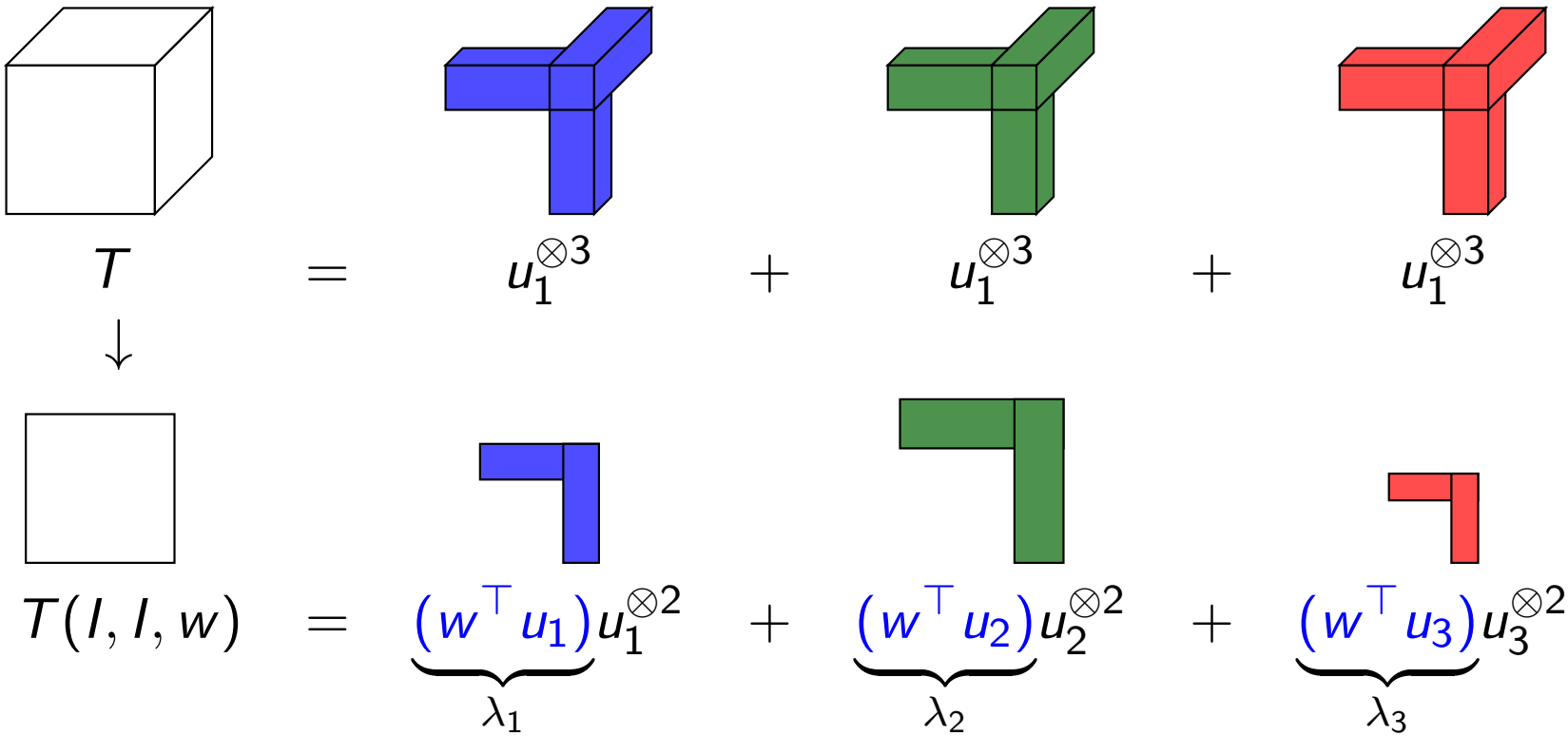

- **Proposal:** Eigen-decomposition on the projected matrix.
- **Return:** recovered eigenvectors, u_i.

Sensitivity of single matrix projection

- **Problem**: Eigendecomposition is very sensitive to the **eigengap**.

$$\text{error in factors} \propto \frac{1}{\min(\text{difference in eigenvalues})}.$$

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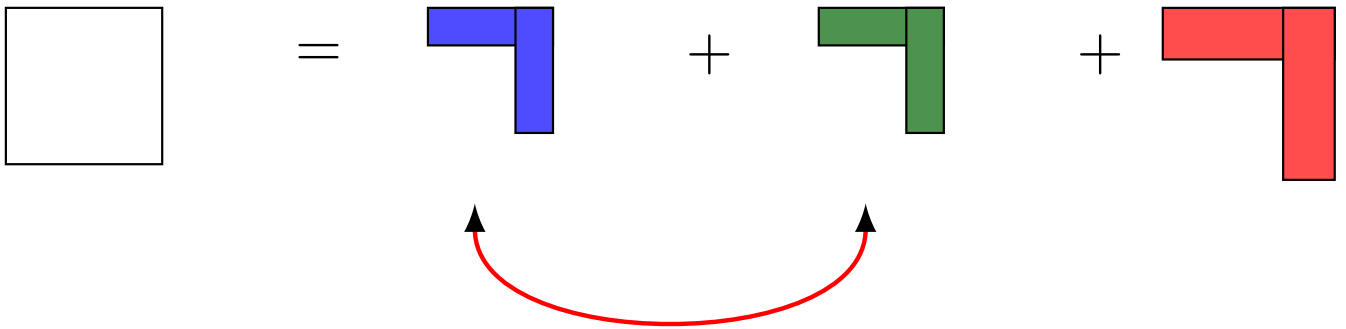

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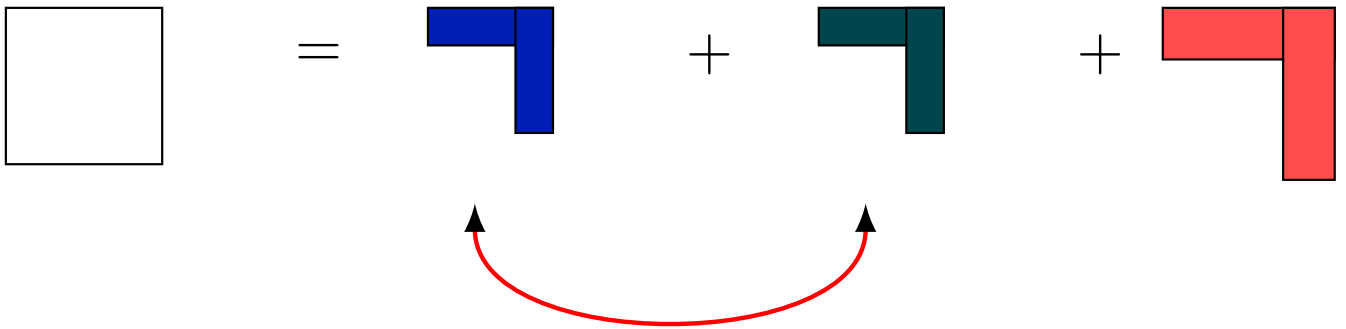
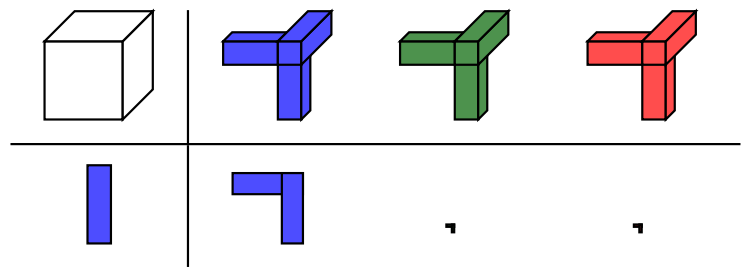


Sensitivity analysis (contd.)



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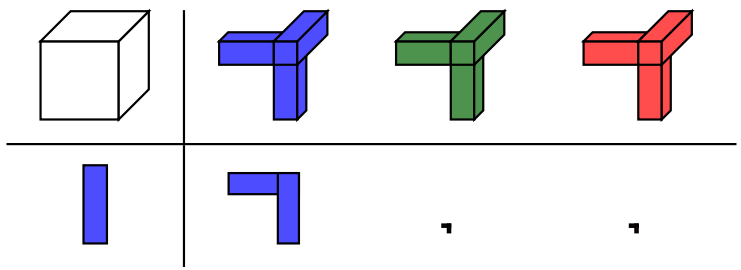

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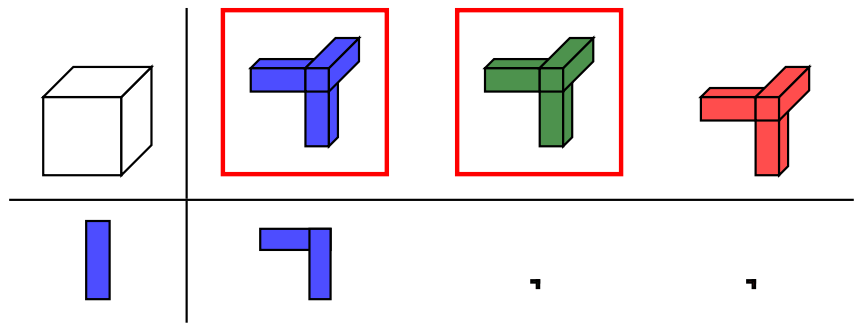

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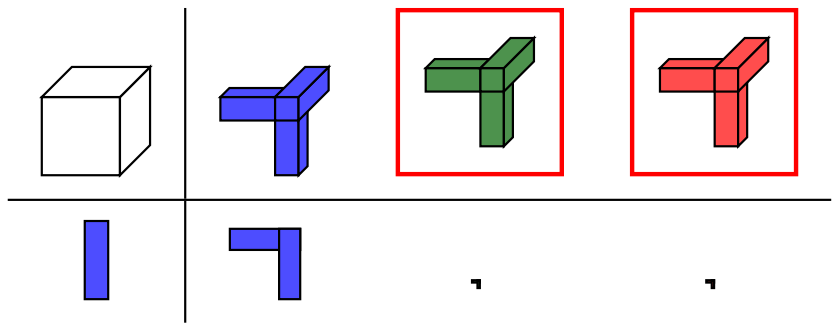

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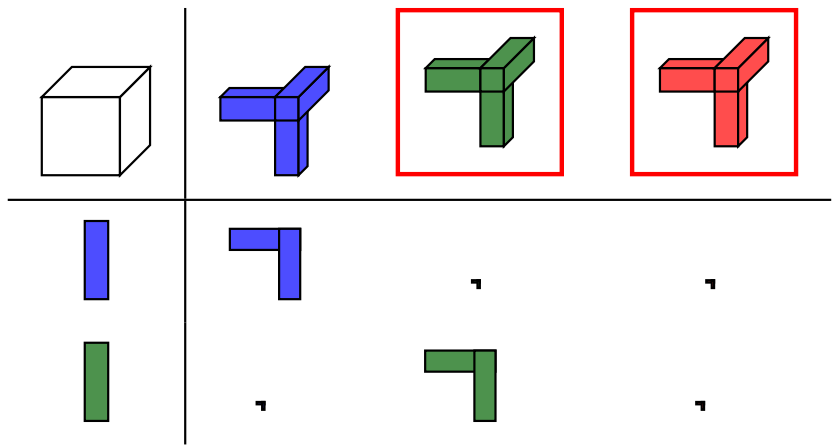

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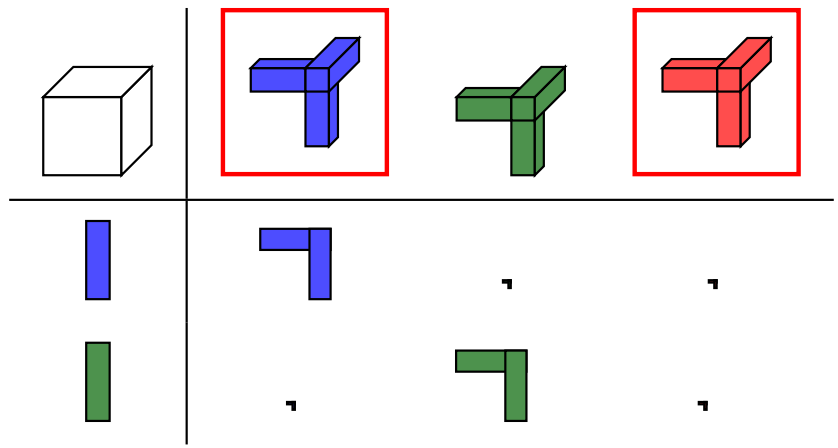

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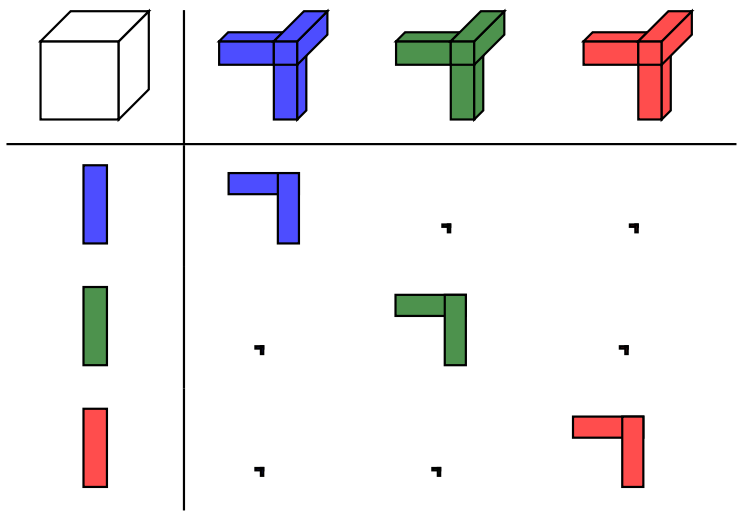

Sensitivity analysis (contd.)

(Cardoso 1994)

- Single matrix factorization:

$$\text{error in factors} \propto \frac{1}{\min \text{diff. in eigenvalues}}.$$

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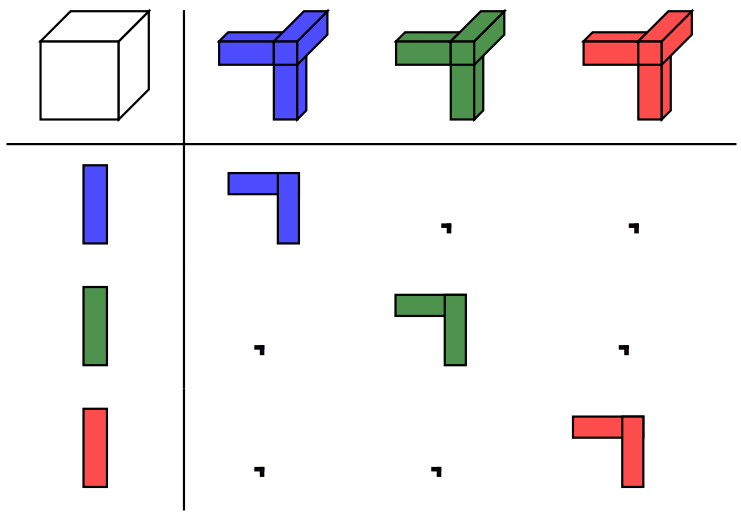
$$\text{error in factors} \propto \frac{1}{\min \text{avg. diff. in eigenvalues}}.$$



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Every coordinate pair needs one good projection (with a large eigengap).

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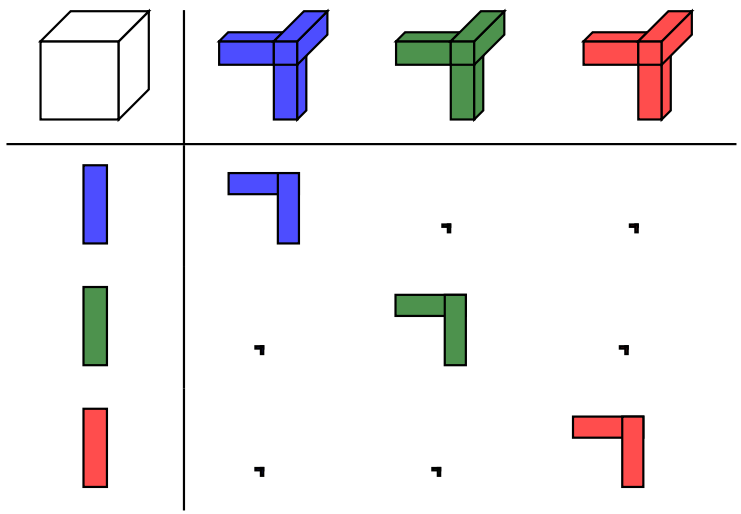

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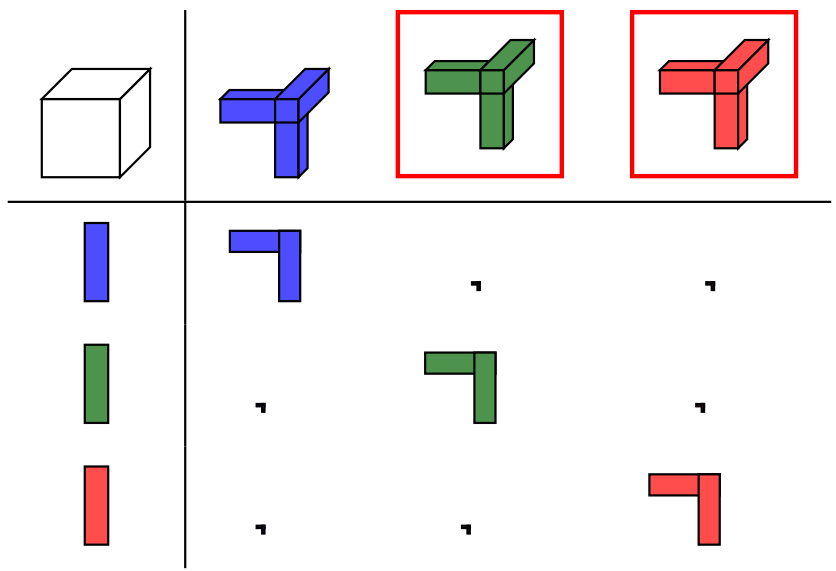

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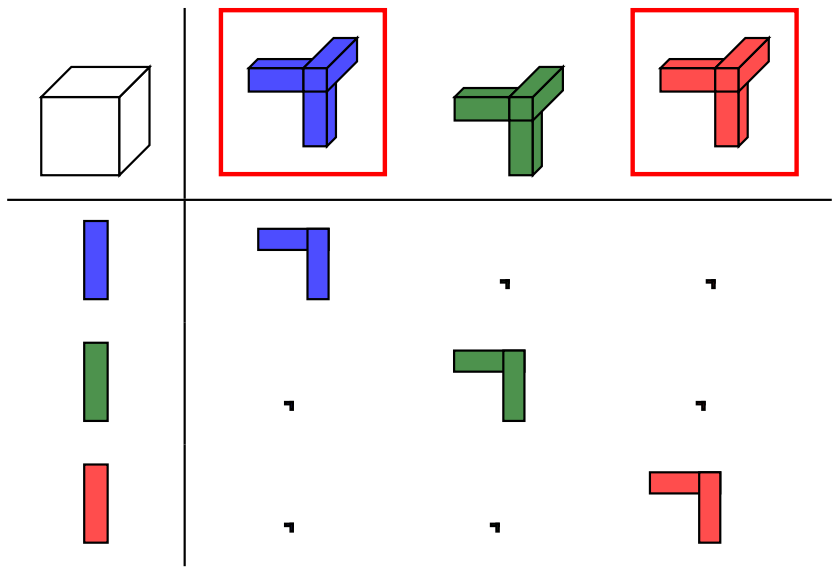

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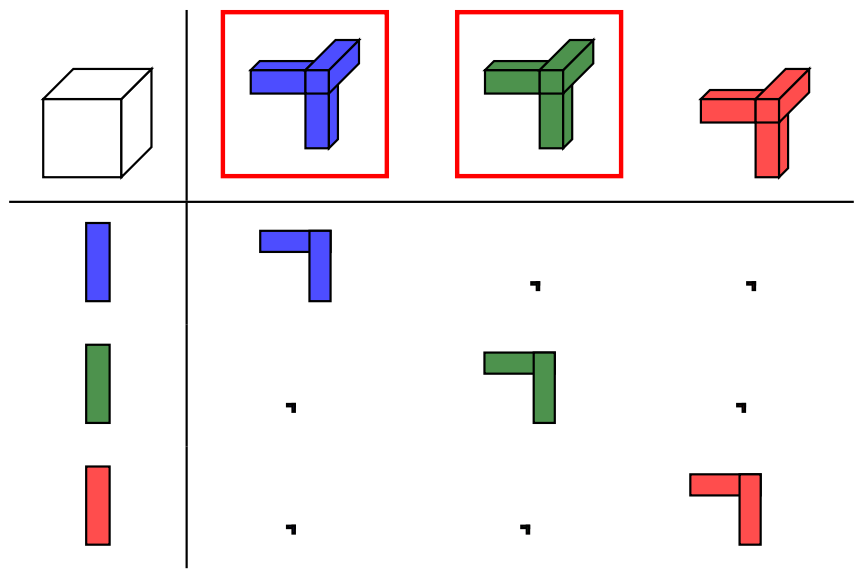
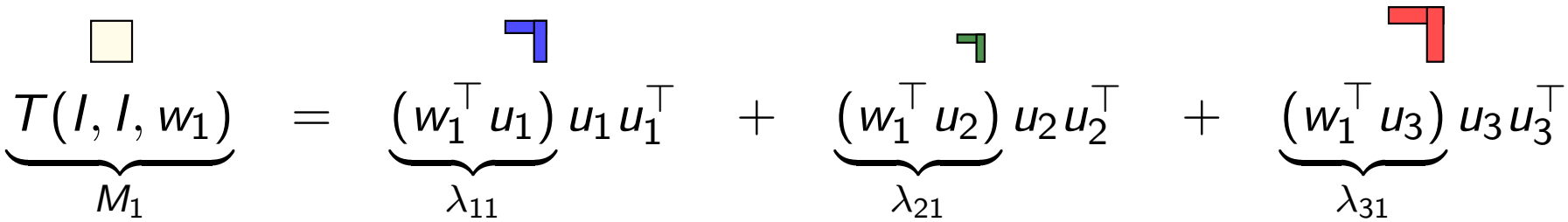


Reduction to simultaneous diagonalization



Reduction to simultaneous diagonalization

$$\underbrace{T(I, I, w_1)}_{M_1} = \underbrace{(w_1^\top u_1)}_{\lambda_{11}} u_1 u_1^\top + \underbrace{(w_1^\top u_2)}_{\lambda_{21}} u_2 u_2^\top + \underbrace{(w_1^\top u_3)}_{\lambda_{31}} u_3 u_3^\top$$

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- **Projections share factors:** can be simultaneously diagonalized.

Simultaneous diagonalization algorithm

- **Algorithm:** Simultaneously diagonalize projected matrices.

$$\widehat{U} = \arg \min_{U: UU^\top = I} \sum_{\ell=1}^{L} \text{off}(U^\top M_\ell U) \qquad \text{off}(A) = \sum_{i \neq j} A_{ij}^2.$$

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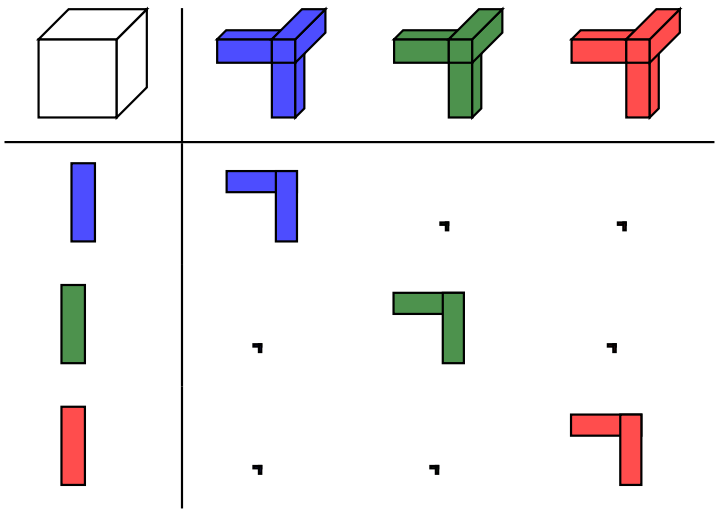
- Optimize using Jacobi angles (Cardoso and Souloumiac 1996).
  - Used widly in the ICA community.
  - Empirically appears to be generically global convergence.

Outline

Oracle and random projections

- **Hypothetically:** "oracle" projections **along the factors** is good.



Oracle and random projections

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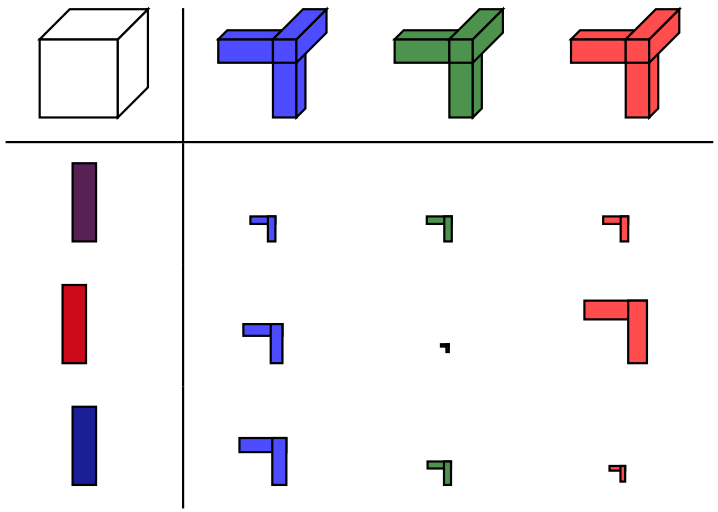
- **Practically:** use projections **along random directions**.



Results: Orthogonal tensor decomposition

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Theorem (Random projections)

Pick $L = \Omega(k \log k)$ *projections* **randomly from the unit sphere**. *Then, with high probability,*

$$\textit{error in factors} \leq O\left(\frac{\sqrt{\|\pi\|_1 \pi_{\max}}}{\pi_{\min}^2} + \sqrt{\frac{d}{L}}\right)\epsilon$$

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Empirical: Random vs. oracle projections

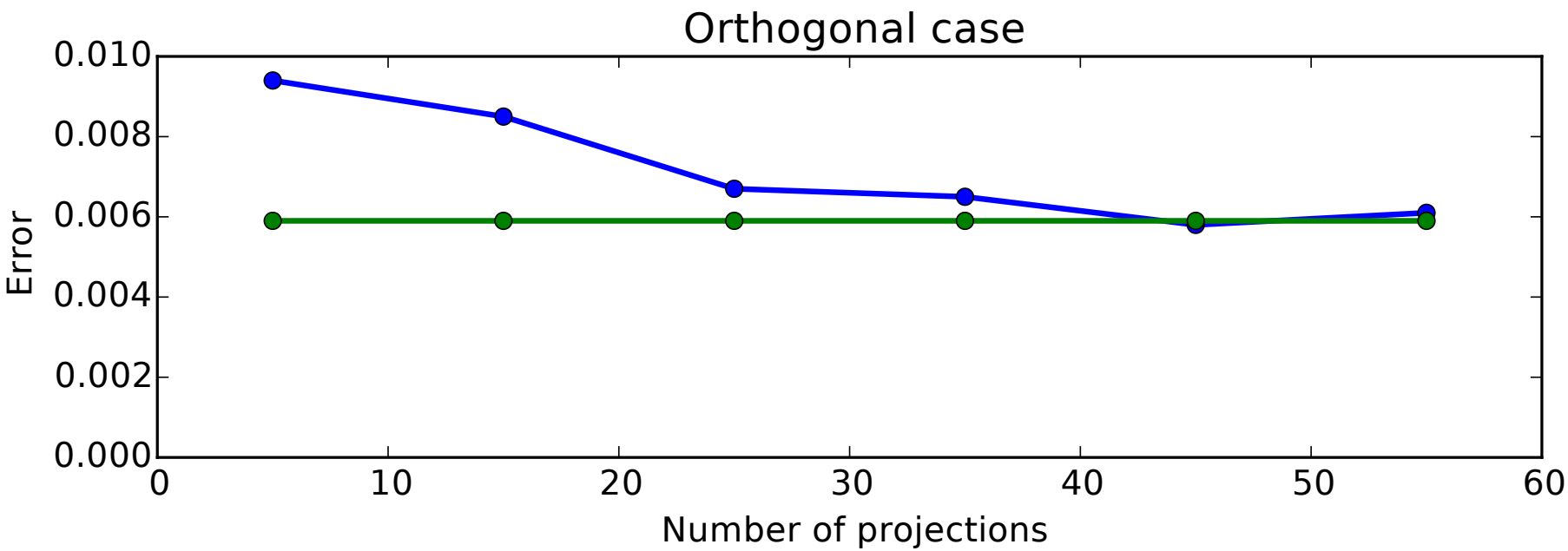
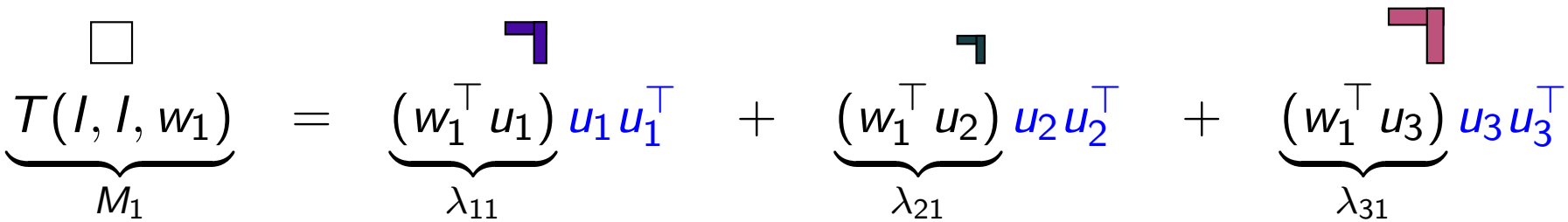


Figure: Comparing random vs. oracle projections ($d = k = 10$, $\epsilon = 0.05$)

Outline

Non-orthogonal simultaneous diagonalization



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- ≥ 2 **matrices have a unique non-orthogonal factorization.**

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- U are not constrained to be orthogonal.
- Optimize using the QR1JD algorithm (Afsari 2006).
  - Only guaranteed to have local convergence.
  - More stable than ALS in practice.

Non-orthogonal simultaneous diagonalization

- **Algorithm:** Simultaneously diagonalize projected matrices.

$$\widehat{U} = \arg\min_{U} \sum_{\ell=1}^{L} \text{off}(U^{-1} M_\ell U^{-\top}) \qquad \text{off}(A) = \sum_{i \neq j} A_{ij}^2.$$

- U are not constrained to be orthogonal.
- Optimize using the QR1JD algorithm (Afsari 2006).
  - Only guaranteed to have local convergence.
  - More stable than ALS in practice.
- Sensitivity analysis due to Afsari 2008

Results: Non-orthogonal tensor decomposition

Theorem (Random projections)

Pick $L = \Omega(k \log k)$ *projections* **randomly from the unit sphere**. *Then, with high probability,*

$$\text{error in factors} \leq O\left(\frac{\|U^{-\top}\|_2^2}{1-\mu^2} \quad \frac{\sqrt{\|\pi\|_1 \pi_{\max}}}{\pi_{\min}^2} \left(1 + \sqrt{\frac{d}{L}}\right) \epsilon \right)$$

where $U = [u_1|\ldots|u_k], \mu = \max u_i^\top u_j$ *and* $\frac{\|U^{-\top}\|_2^2}{1-\mu^2}$ *measures non-orthogonality.*

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Outline

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- Orthogonal tensor methods can factorize non-orthogonal tensors using a whitening transformation (Anandkumar, Ge, Hsu, S. M. Kakade, et al. 2013).
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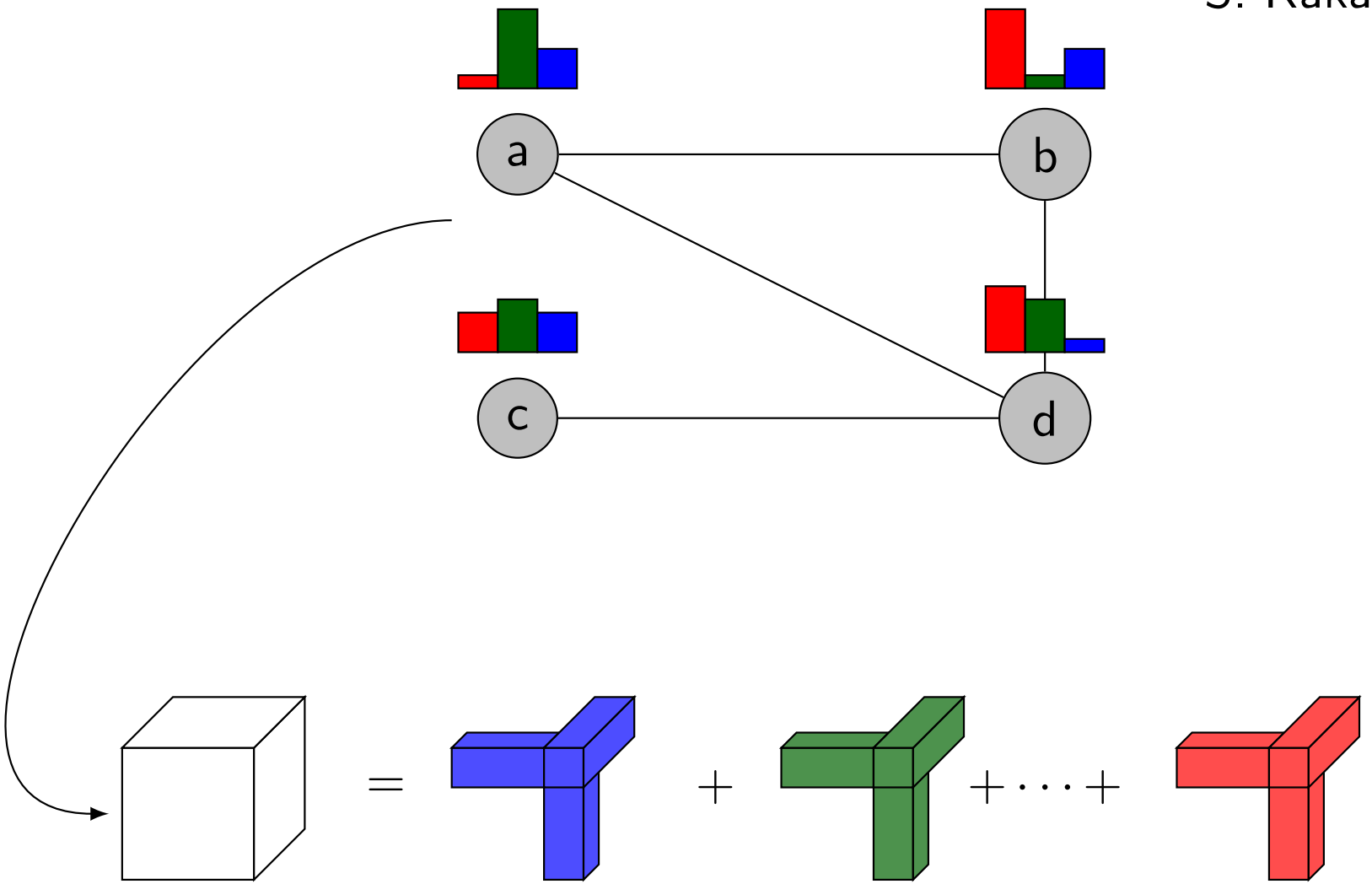
- Simultaneous diagonalization for tensors proposed by Lathauwer 2006.
  - Relies on computing the SVD of a $d^4 \times k^2$ matrix.
- Simultaneous diagonalizations for multiple projections mentioned in Anandkumar, Ge, Hsu, S. M. Kakade, et al. 2013.
  - No analysis presented.

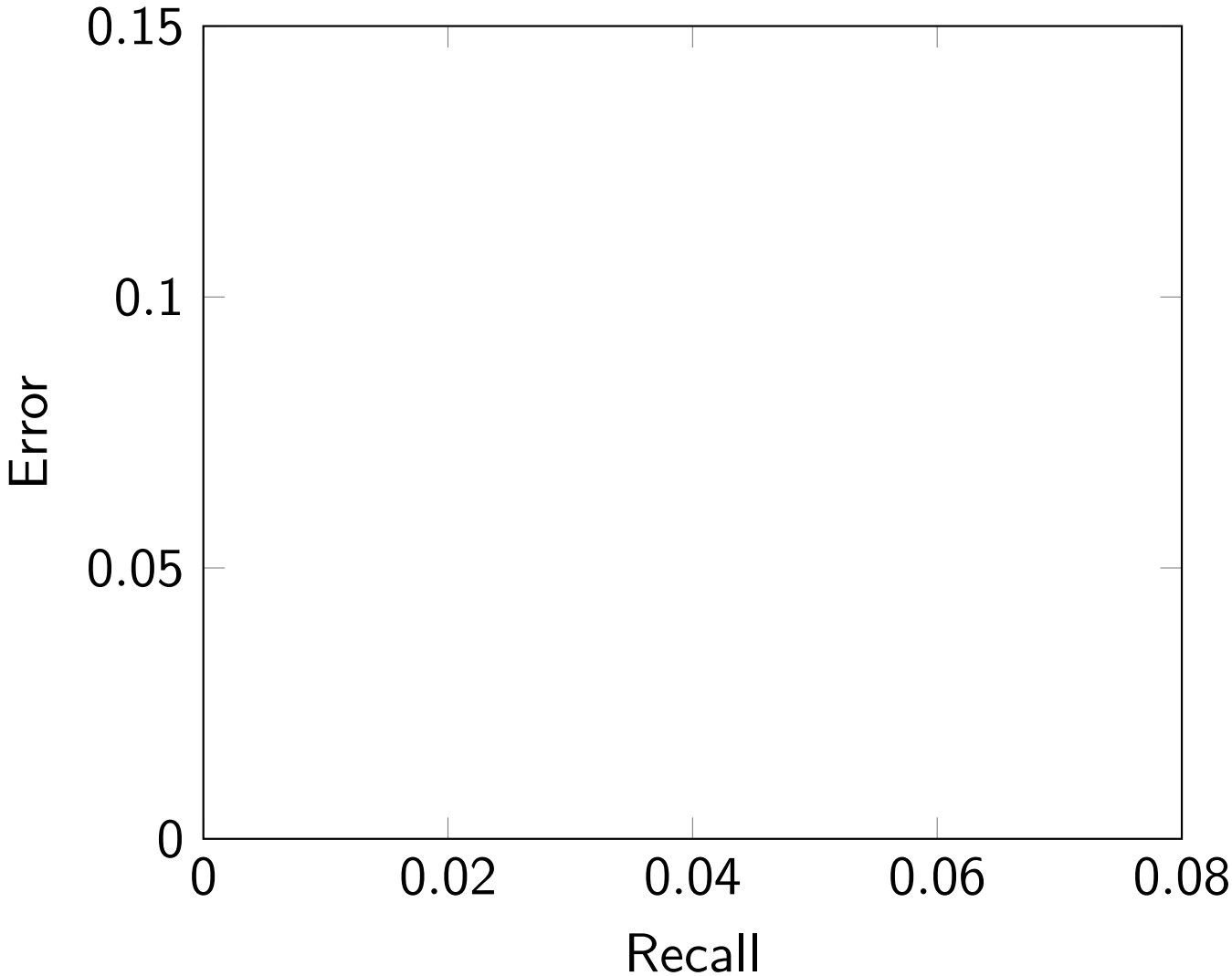
Outline

Community detection

Anandkumar, Ge, Hsu, and S. Kakade 2013



Community detection



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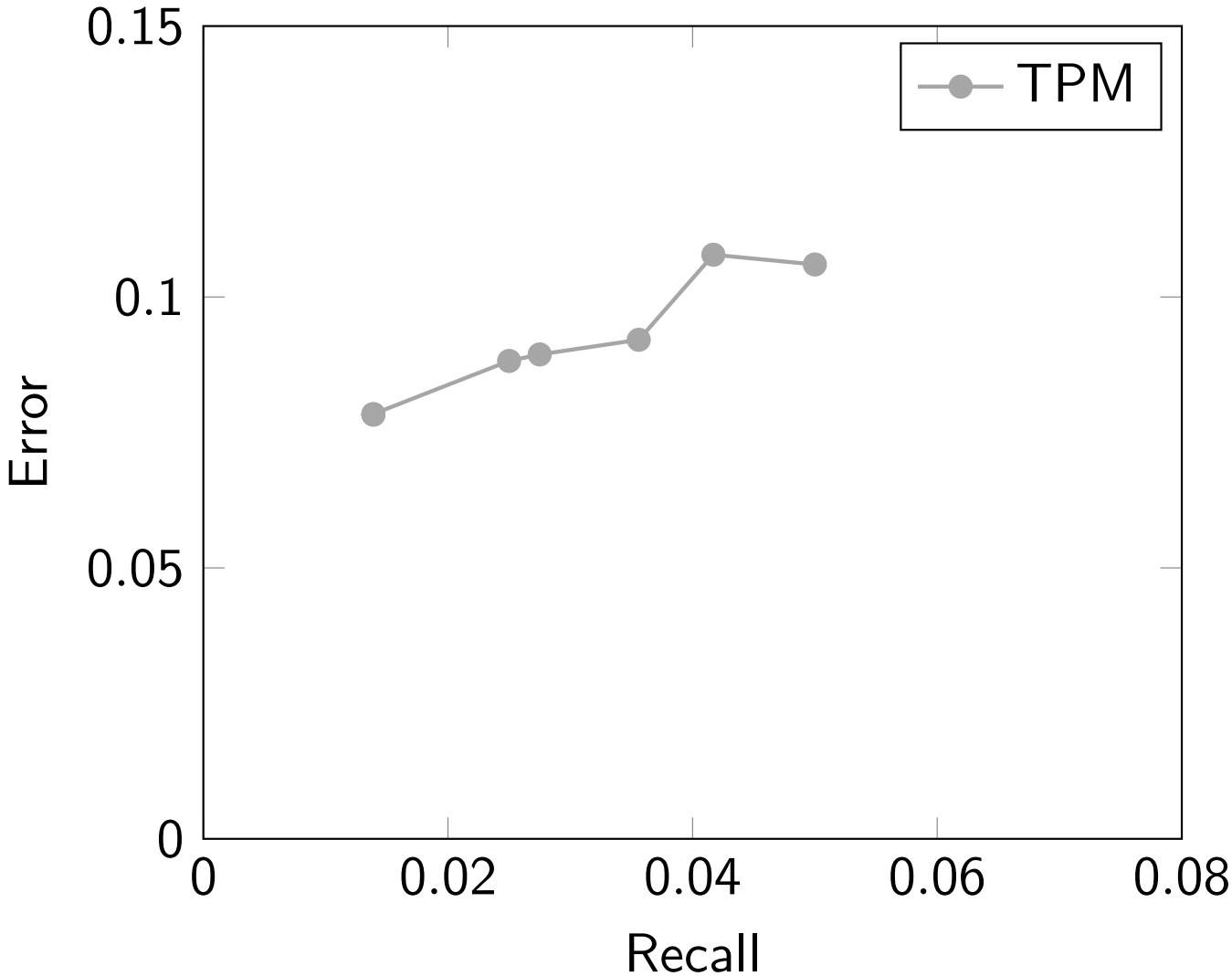

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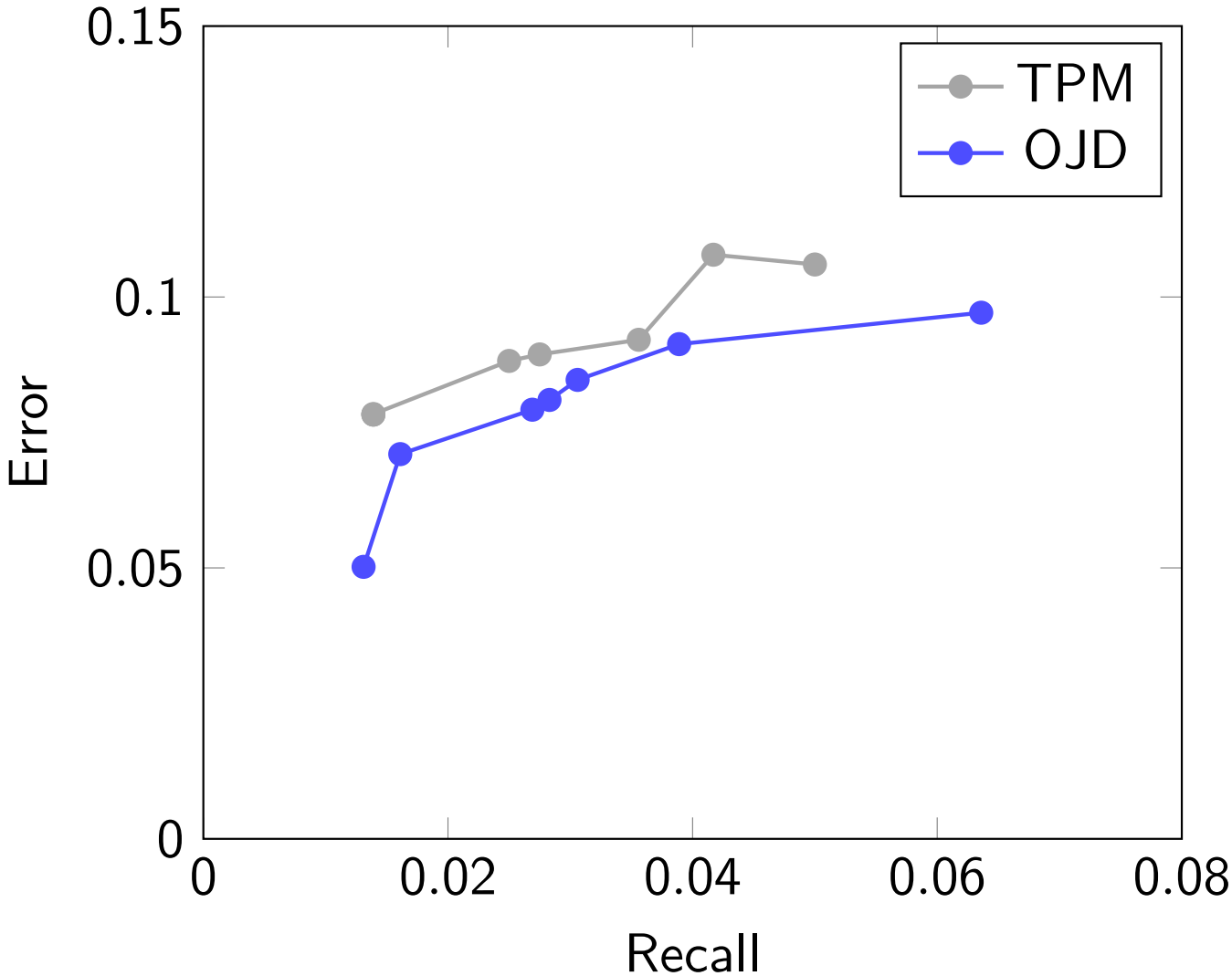
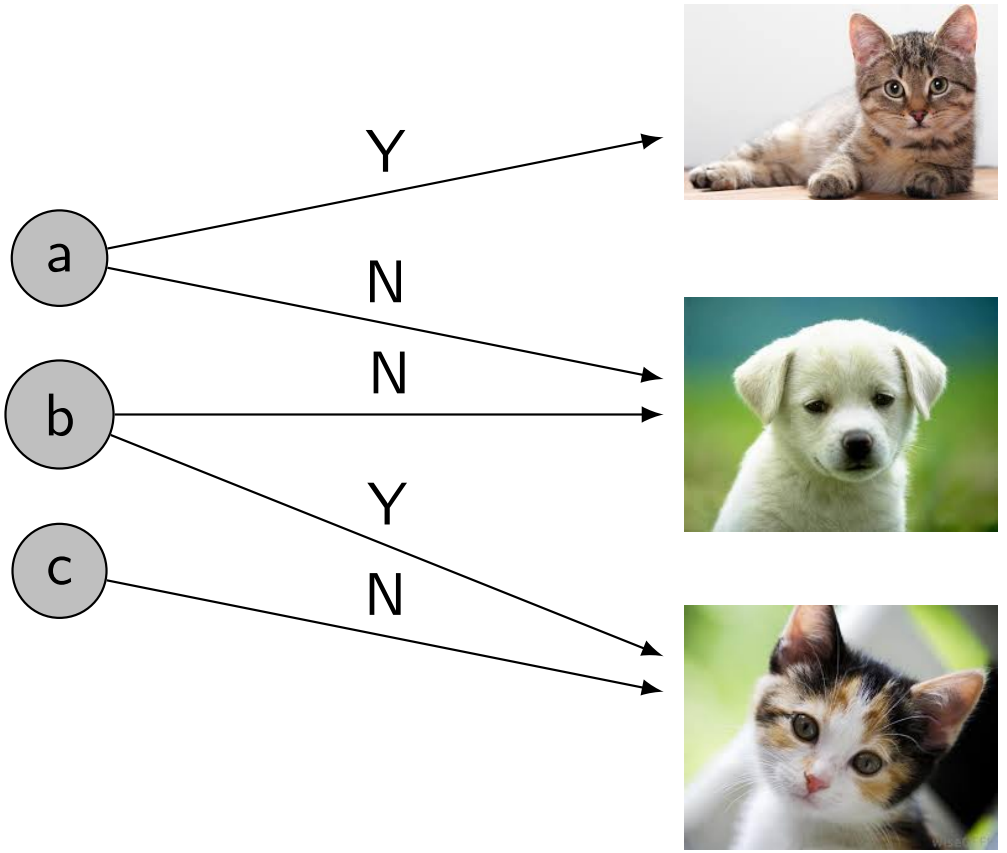


Crowdsourcing

Zhang et al. 2014



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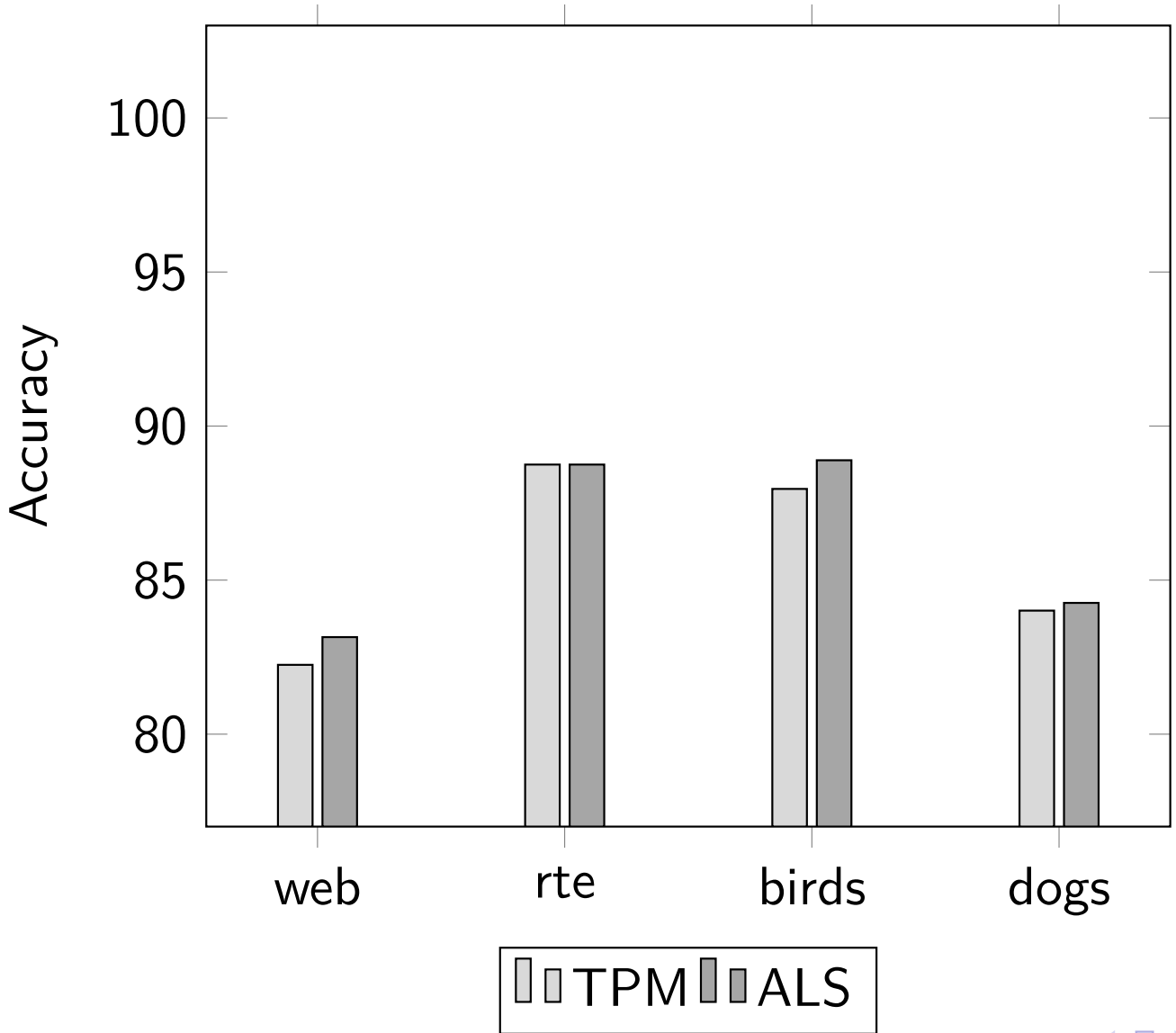

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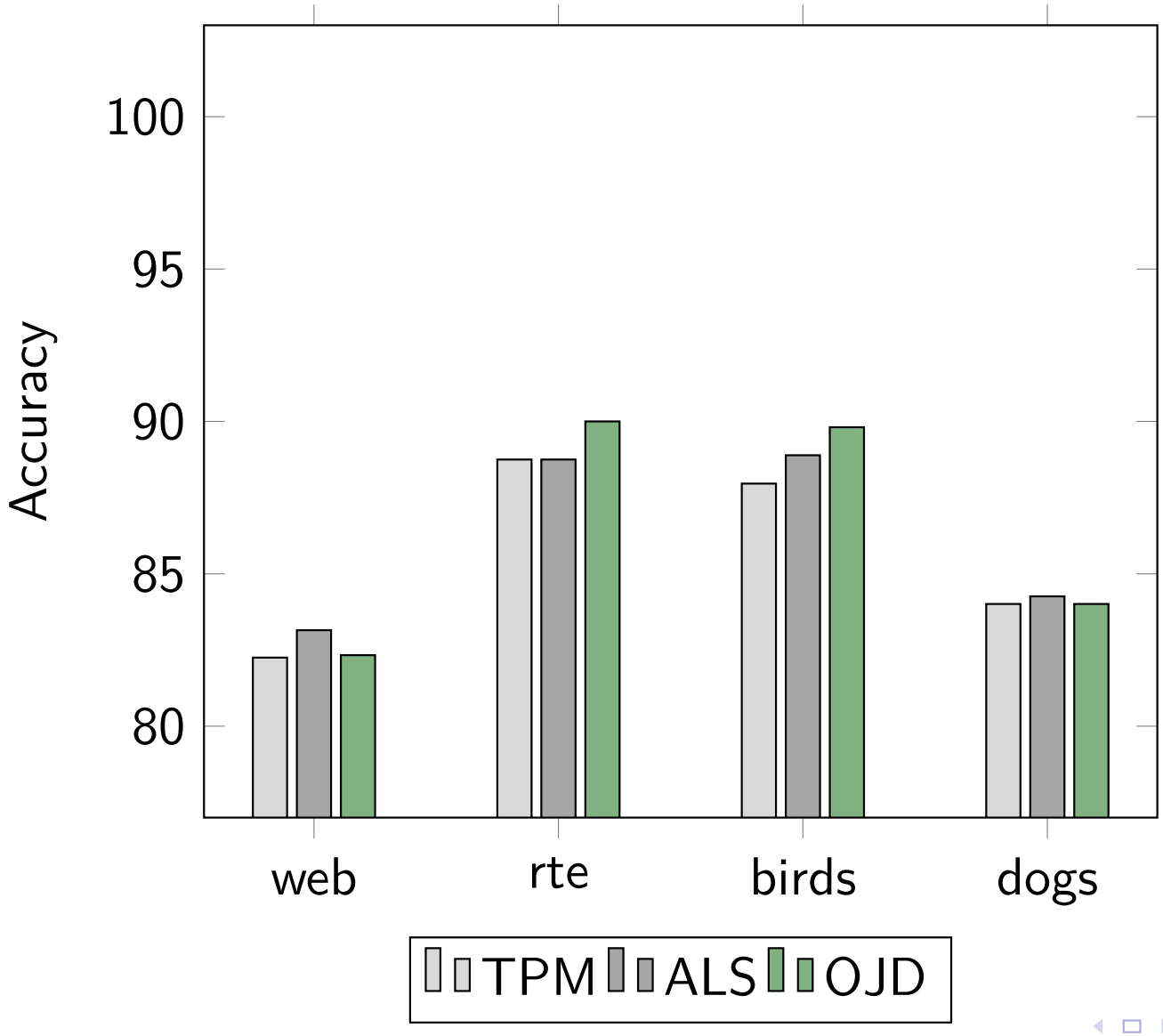

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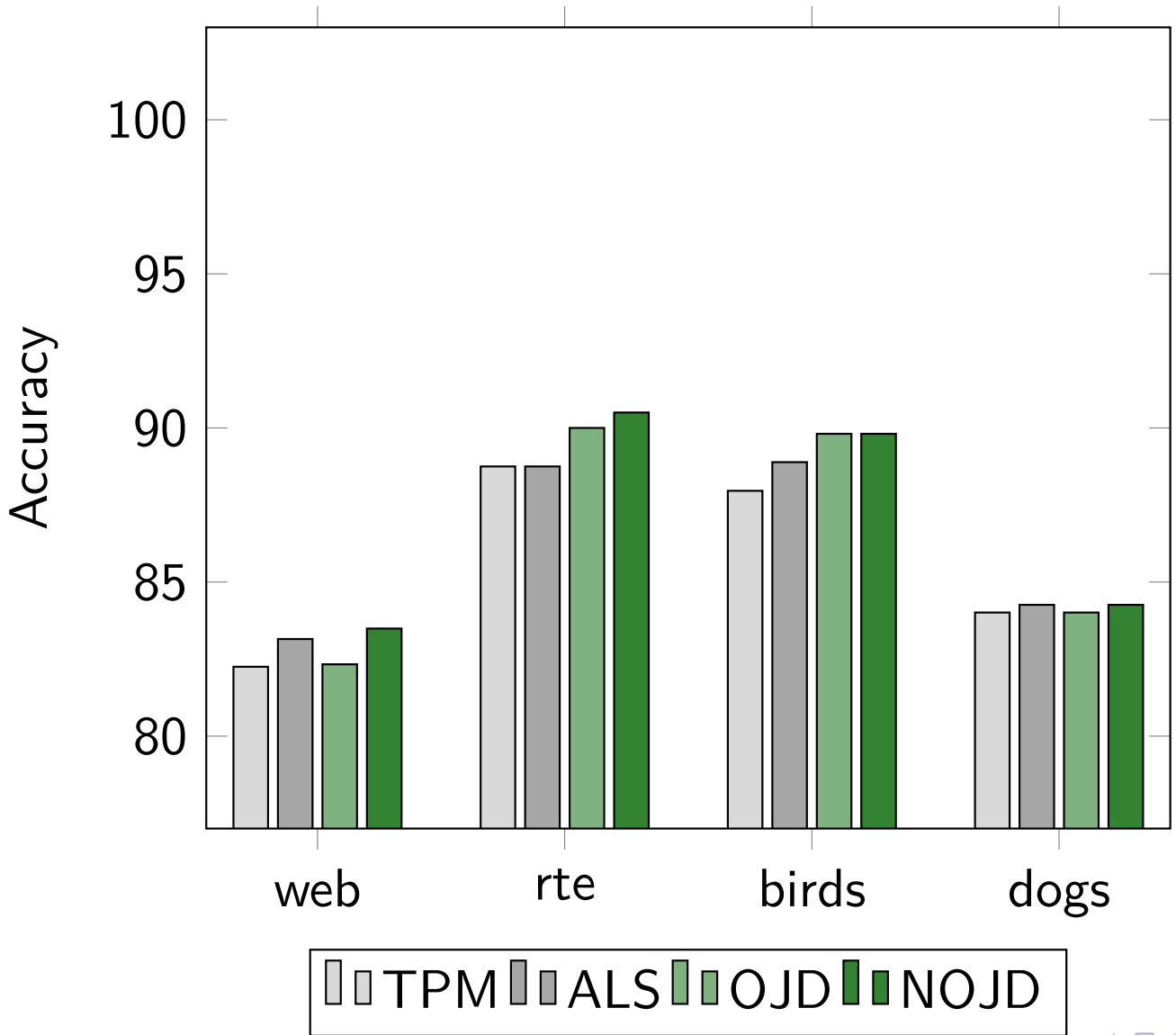

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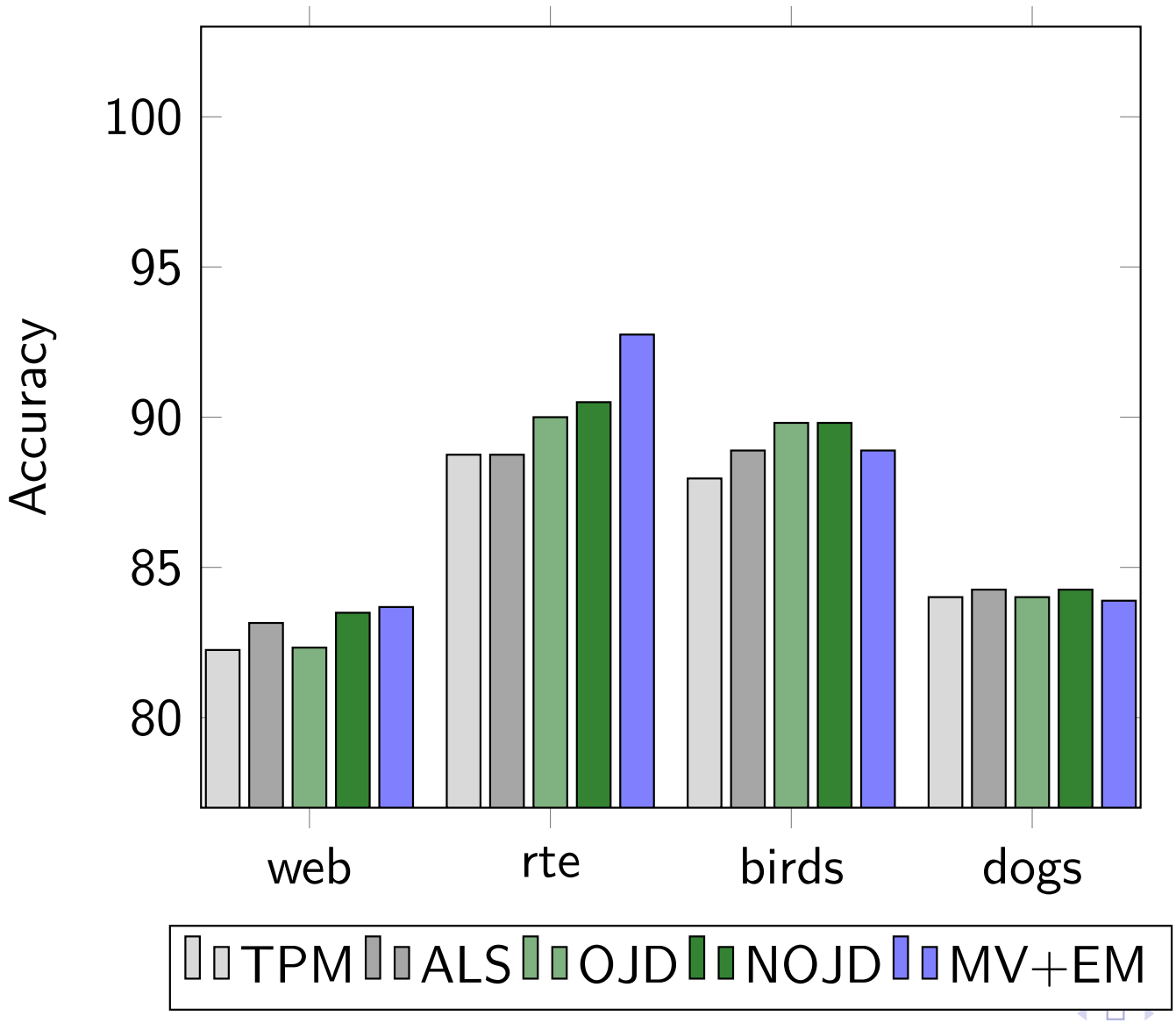
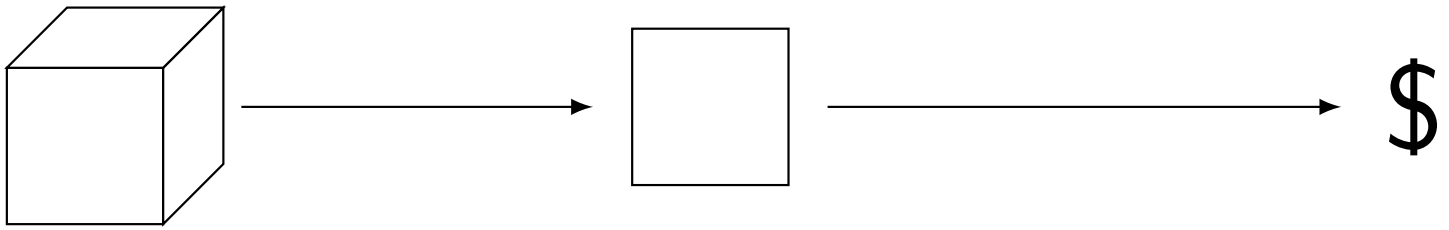


Conclusions



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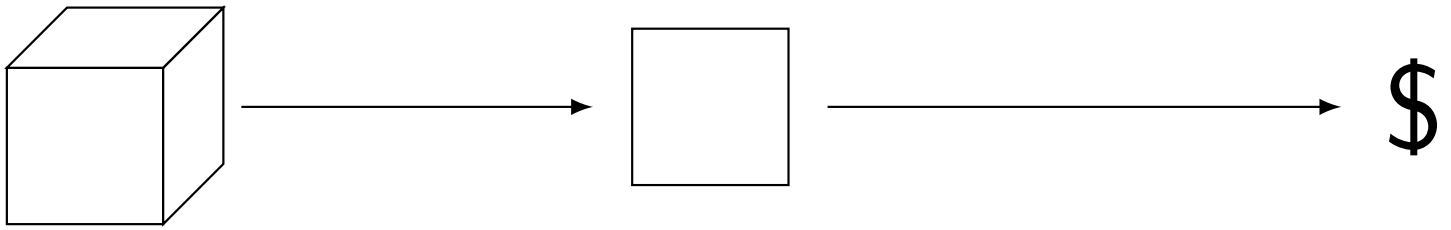

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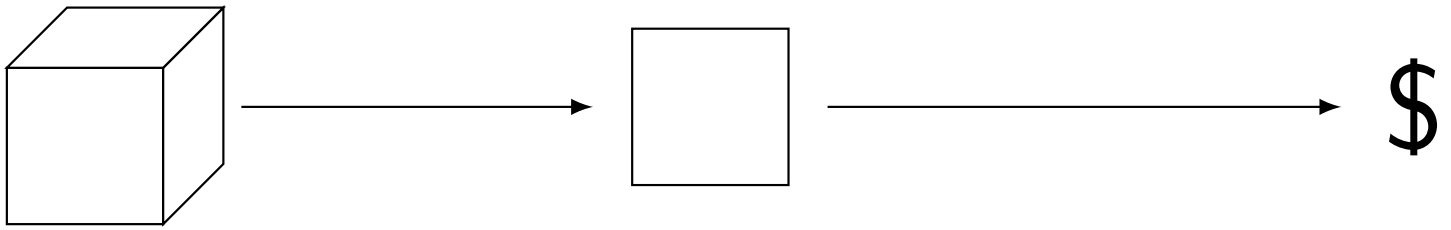

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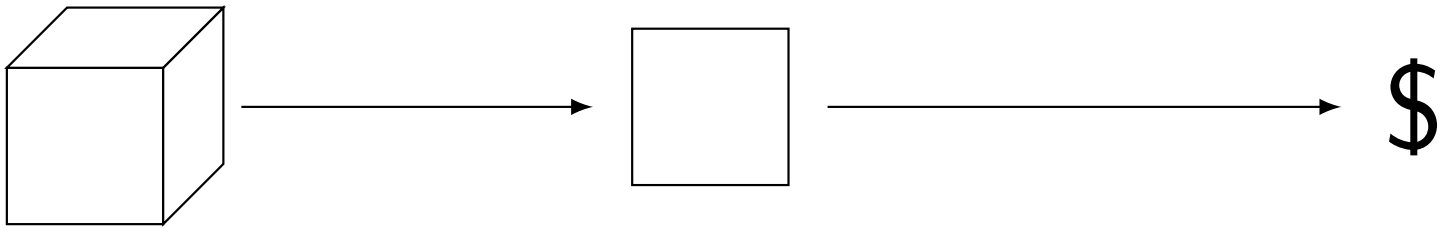

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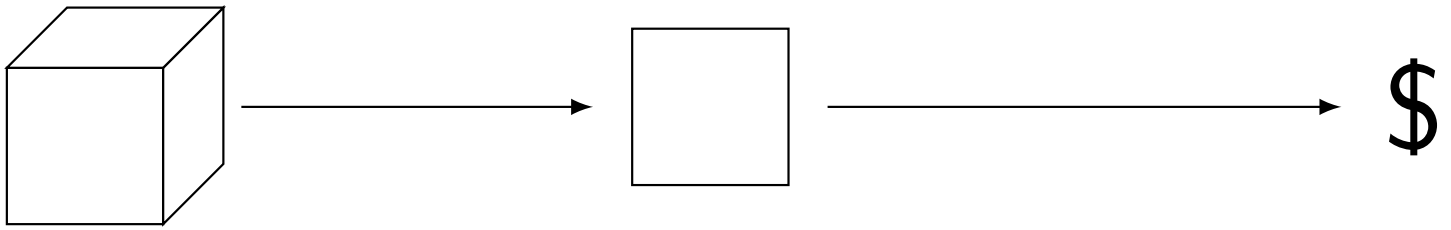

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- **Thanks! Questions?**