

HW 5

Due Nov 25.

1: Sanity check Give *brief* arguments to prove or disprove each of the following.

1. Suppose A is symmetric with eigenvalues $\alpha_1 \geq \dots \geq \alpha_n$ and H is symmetric with eigenvalues $\gamma_1 \geq \dots \geq \gamma_n$. Then the eigenvalues $\mu_1 \geq \dots \geq \mu_n$ of $A + H$ satisfy $\alpha_k + \gamma_n \leq \mu_k \leq \alpha_k + \gamma_1$.
2. Rayleigh-Ritz approximation of the eigenvalues of A using a subspace \mathcal{V} always provides a lower bound on the largest eigenvalue of A .
3. If $\rho_A(x)$ is positive for at least k linearly independent arguments, then A has at least k positive eigenvalues.
4. If $Z \in \mathbb{R}^n$ is a random vector with independent standard normal entries and $A = A^T$, then $E[Z^T A Z] = \text{tr}(A)$. *Note:* You may use without proof the fact, argued in class, that if Q is any fixed orthogonal matrix and Z is a vector of independent standard normals, then QZ is again a vector of independent standard normals.

2: Rayleigh-Ritz and Lanczos Consider the Lanczos decomposition

$$AQ_k = Q_k T_k + \beta_k q_{k+1} e_k^T$$

where $Q_k = [q_1 \ q_2 \ \dots \ q_k]$, the q_j are orthonormal, and T_k is a tridiagonal matrix with diagonal entries $\alpha_1, \dots, \alpha_k$ and off-diagonal entries $\beta_1, \dots, \beta_{k-1}$. Write a routine to compute the Ritz values and Ritz vectors for the subspace spanned by q_1, \dots, q_k , along with the 2-norm of the residual for each Ritz pair. Your code should have the interface

```
function [V,mu,resid] = hw5rr(Q, alpha, beta)
% Compute k normalized Ritz vectors V(:,j), corresponding
% Rayleigh quotients mu(j), and residual norms
%      resid(j) = norm( A*V(:,j)-V(:,j)*mu(j) )
% Input:
%   - Q:      A matrix with k+1 orthonormal columns
%   - alpha:  A vector with k entries
%   - beta:   A vector with k entries
```

3: Ghostly eigenvalues Using the `hw5lanczos` algorithm on the web site, run Lanczos for 100 steps on a random tridiagonal of size 1000 (independent standard normal α_j and β_j) with a random starting vector.

1. Plot the eigenvalues of the leading tridiagonal T_k for $k = 1, \dots, 100$ (and include the plot in your write-up). What do you observe?
2. Augment the code to *explicitly* orthogonalize twice against all previous vectors using modified Gram-Schmidt (optional: do this only when β_k is sufficiently small, a strategy known as selective orthogonalization). Repeat the previous experiment; does the picture change?