HW 4

Due Nov 13.

1: Eigenvalues and singular values Let $J \in \mathbb{R}^{n \times n}$ be the matrix with ones on the first superdiagonal, e.g. in $\mathbb{R}^{4 \times 4}$,

$$J = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and consider the matrix $A = \alpha I - J$.

1. What are the eigenvalue, the column eigenvector, and the row eigenvector of A? What is the condition number of the eigenvalue?

2. What is $\|A^{-1} e_n\|_2$ as a function of n and α?

3. Based on the previous question, give a upper bound on the smallest singular value of A as a function of α and n. Comment on the relation to the eigenvalue.

2: QEP In the analysis of damped vibrations, the *quadratic eigenvalue problem* frequently plays a role:

$$(\lambda^2 I + \lambda B + K)x = 0.$$

where K is typically symmetric and positive definite, and B is typically positive semi-definite.

1. Show how to convert the quadratic eigenvalue problem into a standard eigenvalue problem by adding the equation $y = \lambda x$.

2. Produce a MATLAB function to produce all $2n$ solutions to the quadratic eigenvalue problem. You may call `eig` as a subroutine.

```
function [V,lambda] = hw4qep(B, K)
  % Return eigenvectors V(:,i) associated with eigenvalues lambda(i)
```

3: Solving Sylvester Consider the *Sylvester equation*

$$AX - XB = C$$

where X and C are m-by-n, A is m-by-m, and B is n-by-n. We are going to derive an efficient method for solving Sylvester equations (the *Bartels-Stewart* method).

1. Given the Schur decompositions of A and B, show how $AX - XB = C$ can be transformed into $A'Y - YB' = C'$ where A' and B' are triangular.

2. Describe how to solve the entries of Y one at a time by a process like back substitution. What condition on the eigenvalues of A and B guarantee the system is nonsingular?

3. Use the idea above to implement a Sylvester solver. Your function should have the form

```
function [X] = hw4sylvester(A,B,C)
```

4: Approximate inverses Suppose the eigenvalues of a symmetric matrix A all lie in an interval $[\alpha, \beta]$ that does not contain zero, and that p is some polynomial such that $|p(z) - 1/z| \leq C$ for any $z \in [\alpha, \beta]$. Argue that for any b,

$$\|A^{-1}b - p(A)b\|_2 \leq C\|b\|_2.$$