HW 3

Due: Monday, Oct 7.

Remember that you may (and should!) talk about the problems amongst yourselves, or discuss them with me, providing attribution for any good ideas you might get – but your final write-up should be your own.

1: Hessenberg LU. A matrix $H \in \mathbb{R}^{n \times n}$ is *upper Hessenberg* if $H_{i,j} = 0$ for $i > j+1$. That is, anything below the first subdiagonal must be zero. Without using MATLAB's `lu`, write an $O(n^2)$ function to compute $PH = LU$ using Gaussian elimination with partial pivoting. Note that the Schur complements will always remain upper Hessenberg: why? Your function should have the signature

```
function [P,L,U] = hw3lu(H)
```

2: Elementary, upon reflection. Complete the following routine:

```
function [u] = hw3reflect(v,w)
% Find a reflector H = I-2*u*u' s.t. w and H*v are parallel.
% Assume v and w are nonzero column vectors of equal length.
```

3: Alternate orthogonalization. Recall that any symmetric positive definite matrix M defines an inner product $\langle x, y \rangle_M = y^T M x$ and an associated norm $\|x\|_M^2 = x^T M x$. We can associate with this an alternate notion of the QR decomposition:

1. Write a MATLAB function `[W,R] = hw3wr(A,M)` that computes the factorization
$$A = WR, \quad W^T M W = I.$$
Hint: Use QR factorization of A and Cholesky factorization of M.

2. Write the matrix M such that if $p(x) = \sum_{i=0}^{d} a_i x^i$ and $q(x) = \sum_{j=0}^{d} b_j x^j$ are polynomials, then
$$a^T M b = \int_{-1}^{1} p(x) q(x) \, dx$$
where $a = \begin{bmatrix} a_0 & a_1 & \dots & a_d \end{bmatrix}$ is the coefficient vector for p, and b is the coefficient vector for q. Using the `hw3wr` command in the first part

of the problem to compute the *normalized Legendre polynomials* up to degree 5; these are polynomials q_k such that the degree of P is k and

$$\int_{-1}^{1} q_k(x) q_j(x)\, dx = \begin{cases} 1, & k = j \\ 0, & \text{otherwise,} \end{cases}$$

4: Refined tastes. Consider the bordered linear system

$$\begin{bmatrix} A & b \\ b^T & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix}.$$

where

$$A = \begin{bmatrix} 1 & 1-h \\ 1-h & 1 \end{bmatrix}, \qquad f = \begin{bmatrix} 1 \\ -1 \end{bmatrix},$$

$$b = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad g = 0.5,$$

$$c = 1$$

and $h = 10^{-14}$.

1. Solve the bordered system both by forming the whole 3-by-3 matrix and solving using MATLAB's backslash and by doing 2-by-2 block Gaussian elimination. What are the normwise relative residual errors for the two calculated results?

2. In the block 2-by-2 solve, what is the normwise relative residual for the step involving a solve with A?

3. What are the condition numbers of A and of the bordered system, respectively? Argue that the relative error in the x computed by block 2-by-2 elimination is consistent with the bounds based on the relative residual and the condition number.

4. For the 2-by-2 block approach, perform one step of iterative refinement on the bordered system. What is the residual residual error for this new solution?

Note: This problem was inspired by the article "Block Elimination with One Refinement Solves Bordered Systems Accurately" (Govaerts and Pryce, BIT, vol 30, pp. 490–507, 1990).