HW 2

Due: Monday, Sep 23.

Remember that you may (and should!) talk about the problems amongst yourselves, or discuss them with me, providing attribution for any good ideas you might get – but your final write-up should be your own.

1: Approximately fun. Let $B(s) = (A + sE)^{-1}$, where A is invertible.

1. Argue that $B(s)$ is well-defined when $|s| < 1/\|A^{-1}E\|_2$.
2. Write a MATLAB fragment to evaluate $\bar{B}(s) = B(0) + sB'(0)$; your code should *not* use the `inv` function, but instead should employ forward and backward slash operations.
3. For `A = rand(10); E = rand(10)`, plot $\|B(s) - \bar{B}(s)\|_F$ against s on a log-log plot for $s = 10^{-1}$ down to $s = 10^{-16}$. What would you expect to see in exact arithmetic?
4. What actually happens in floating point? Why?

2: Oh, Schatten. Let $A \in \mathbb{R}^{n\times n}$ with SVD $A = U\Sigma V^T$. The *Schatten p-norm* is the p norm of the vector of singular values:

- $\|A\|_2 = \max_i \sigma_i$ is the Schatten ∞-norm
- $\|A\|_F = \sqrt{\sum_i \sigma_i^2}$ is the Schatten 2-norm
- $\|A\|_* = \sum_i \sigma_i$ is the Schatten 1-norm

The Schatten 1-norm is also called the nuclear norm or trace norm; it appears in various compressed sensing applications.

1. Show: for any $A, B \in \mathbb{R}^{n\times n}$, we have $\operatorname{tr}(AB) = \operatorname{tr}(BA)$.
2. Argue that the diagonal elements of $V^T U$ are at most 1.
3. Show that $\operatorname{tr}(A) \leq \|A\|_*$. *Hint*: SVD plus parts 1–2.
4. Show $\|A + B\|_* \leq \|A\|_* + \|B\|_*$. *Hint:* SVD of $A + B$ plus parts 1–3.
5. Argue that the nuclear norm is not induced by any vector norm. *Hint:* What is the norm of the identity in any operator norm?

Note: Shoot for short answers – my arguments are all one or two lines.

3: Consider conditioning. Let $A \in \mathbb{R}^{n \times n}$ be a symmetric matrix and $u \in \mathbb{R}^n$ a vector, and define $\phi = u^T A u$. Define norm-wise condition numbers κ_u and κ_A that describe the sensitivity of ϕ to perturbations in u and A.

4: Run for the bordered system. Complete the following:

```
function [X] = hw2p4(solveA,C,B)
% Compute X = K\B, where K = [A, C; C', 0].
% Inputs:
%   solveA - a function s.t. Y = solveA(X) solves A\X
%   C      - an m-by-p bordering matrix
%   B      - a matrix of size (m+p)-by-q
```

Your function should solve $p + q$ linear systems using `solveA`. Apart from the calls to `solveA`, what is the asymptotic complexity of the solver?