Final

Due: Wednesday, Dec 18.

Choose any ten of the following problems You may use the textbook or other references (with citation). You may discuss problems with me, but not with each other. Some of these are pretty simple (and others less so), so don't psych yourself out. Please be brief! And please feel free to use MATLAB to computationally sanity-check yourself.

1. Show: If $PA = LU$ is an LU factorization and $A = QR$ is a QR factorization of the same square matrix A, then $|\prod_i u_{ii}| = |\prod_i r_{ii}|$.

2. Show: Solving the linear system

$$\begin{bmatrix} I & A \\ A^T & -\mu I \end{bmatrix} \begin{bmatrix} r \\ x \end{bmatrix} = \begin{bmatrix} b \\ 0 \end{bmatrix}$$

   is equivalent to solving a regularized least squares problem. Please also state the corresponding objective function.

3. Show: If a real symmetric matrix A has at least one positive diagonal entry, it must have at least one positive eigenvalue.

4. If $A \in \mathbb{R}^{n\times n}$ is symmetric and nonsingular and $A = U\Sigma V^T$ is the SVD, then the inertia of A can be computed from U and V alone. Briefly describe how.

5. For any $\epsilon > 0$, show how to construct a matrix $T(\epsilon) \in \mathbb{R}^{2\times 2}$ with real eigenvalues $0 < \lambda_1 \leq \lambda_2$ such that $\sigma_2 \leq \epsilon\lambda_1$.

6. Consider the block 2-by-2 matrix

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}.$$

   Show M is nonsingular if A and D are invertible and for some operator norm,

$$\|A^{-1}\|\|D^{-1}\|\|B\|\|C\| < 1.$$

7. Consider a nonsymmetric block 2-by-2 matrix

$$A = \begin{bmatrix} \Lambda_1 & E \\ F & \Lambda_2 \end{bmatrix},$$

where Λ_1 and Λ_2 are diagonal. Show that the eigenvalues of A must lie in the union of the disks of radius $\rho = \sqrt{\|E\|_2 \|F\|_2}$. You may use the result of the previous statement without proof.

8. Suppose $A = D - K$ is strictly diagonally dominant with the diagonal part D and off-diagonal part K both elementwise non-negative. Let b be an elementwise non-negative vector, and let $x = A^{-1}b$. Show that Jacobi iteration starting at $x^{(0)} = 0$ converges to b from below (i.e. $x^{(k)} \leq x^{(k+1)} \leq x$ elementwise for each k).

9. Suppose $A \in \mathbb{R}^{n \times n}$ is a symmetric tridiagonal matrix with constant diagonal and off-diagonal entries α and β. Write the eigenvalues of A in closed form. (*Hint:* reduce to the model problem in 1D.)

10. For $A \in \mathbb{R}^{m \times n}$ with $m > n$ and $b \in \mathbb{R}^m$, the iteration

$$x^{(k+1)} = x^{(k)} + \alpha A^T (b - Ax^{(k)})$$

converges linearly for a range of positive α. Describe the range of positive α values leading to convergence and the fixed point to which the iteration converges.

11. Suppose $A \in \mathbb{R}^{n \times n}$ is symmetric positive semi-definite and rank r, and consider the regularized linear system $(A + \mu I)x = b$. Argue that $x \in \mathcal{K}_{r+1}(A, b)$.