Week 9: Wednesday, Oct 17

A brief admonishment

One of the great benefits of being a numerical mathematician is that you can test your ideas on the computer. Even if coming up with problems that thoroughly test hard cases where some algorithm might fail is hard, a few randomly chosen tests is often enough to smoke out all sorts of problems, from programming errors to more fundamental misunderstandings.

So in HW 3, I was disappointed at how many submissions were clearly never sanity-checked. For the first two problems, some of you produced codes that failed to run at all, and many of you wrote codes that produced clearly incorrect results (as determined by comparing, for a randomly constructed example, the x vector from the equivalent $O(n^3)$ computation that I gave you in the comments). For the third problem, the simple check

```
A = randn(3,2);
I = eye(3);
Z = zeros(2);
B = [I, A; A', Z];
s  = svd(A)
sB = svd(B)
cond(B)
```

could have been used to sanity check the singular value computations, and the ultimate condition number calculation. For the fourth problem, a plot of the difference between $\cos(x)$ and the computed polynomial would have caught most of the computational errors.

More generally, I would encourage you to do numerical experiments to try to see the patterns when you're figuring out the types of questions you see in the homework for this class. Often, working out the details on a small example problem can quickly give you a hypothesis that you can generalize, while starting in search of a general hypothesis before doing any examples takes much longer.

Needless to say, I hope you will take this general advice to heart as you work on the midterm. At the very least, please make sure you test your submissions.

Perturbing Gershgorin

Last time, we showed the Gershgorin theorem: the eigenvalues of a matrix A all lie inside the Gershgorin disks

$$G_j = \{z : |a_{jj} - z| \leq \sum_{i \neq j} |a_{ij}|\},$$

and any connected component of $\cup_j G_j$ consisting of exactly m disks will contain exactly m eigenvalues of A.

Now, let us consider the relation between the Gershgorin disks for a matrix A and a matrix $\hat{A} = A + F$. It is straightforward to write down the Gershgorin disks $\hat{G}_j$ for $\hat{A}$:

$$\hat{G}_j = \mathcal{B}_{\hat{\rho}_j}(\hat{a}_{jj}) = \{z \in \mathbb{C} : |a_{jj} + e_{jj} - z| \leq \hat{\rho}_j\} \text{ where } \hat{\rho}_j = \sum_{i \neq j} |a_{ij} + f_{ij}|.$$

Note that $|a_{jj} + e_{jj} - z| \geq |a_{jj} - z| - |f_{jj}|$ and $|a_{ij} + f_{ij}| \leq |a_{ij}| + |f_{ij}|$, so

$$(1) \qquad \hat{G}_j \subseteq \mathcal{B}_{\rho_j + \sum_j |f_{ij}|}(a_{jj}) = \left\{ z \in \mathbb{C} : |a_{jj} - z| \leq \rho_j + \sum_i |f_{ij}| \right\}.$$

We can simplify this expression even further if we are willing to expand the regions a bit:

$$(2) \qquad \hat{G}_j \subseteq \mathcal{B}_{\rho_j + \|F\|_1}(a_{jj}).$$

The Bauer-Fike theorem

We now apply Gershgorin theory together with a carefully chosen similarity to prove a bound on the eigenvalues of $A+E$ where E is a finite perturbation. This will lead us to the *Bauer-Fike* theorem.

The basic idea is as follows. Suppose that A is a diagonalizable matrix, so that there is a complete basis of column eigenvectors V such that

$$V^{-1}AV = \Lambda.$$

Then we $A + F$ has the same eigenvalues as

$$V^{-1}(A + F)V = \Lambda + V^{-1}FV = \Lambda + \tilde{F}.$$

Now, consider the Gershgorin disks for $\Lambda + \tilde{F}$. The crude bound (2) tells us that all the eigenvalues live in the regions

$$\bigcup_j \mathcal{B}_{\|\tilde{F}\|_1}(\lambda_j) \subseteq \bigcup_j \mathcal{B}_{\kappa_1(V)\|F\|_1}(\lambda_j).$$

This bound really is crude, though; it gives us disks of the same radius around all the eigenvalues λ_j of A, regardless of the conditioning of those eigenvalues. Let's see if we can do better with the sharper bound (1).

To use (1), we need to bound the absolute column sums of $\tilde{F}$. Let e represent the vector of all ones, and let e_j be the jth column of the identity matrix; then the jth absolute column sums of $\tilde{F}$ is $\phi_j \equiv e^T|\tilde{F}|e_j$, which we can bound as $\phi_j \leq e^T|V^{-1}||F||V|e_j$. Now, note that we are free to choose the normalization of the eigenvector V; let us choose the normalization so that each row of $W^* = V^{-1}$. Recall that we defined the angle θ_j by

$$\cos(\theta_j) = \frac{|w_j^* v_j|}{\|w_j\|_2 \|v_j\|_2},$$

where w_j and v_j are the jth row and column eigenvectors; so if we choose $\|w_j\|_2 = 1$ and $w_j^* v_j = 1$ (so $W^* = V^{-1}$), we must have $\|v_j\|_2 = \sec(\theta_j)$. Therefore, $\||V|e_j\|_2 = \sec(\theta_j)$. Now, note that $e^T|V^{-1}|$ is a sum of n rows of Euclidean length 1, so $\|e^T|V^{-1}|\|_2 \leq n$. Thus, we have

$$\phi_j \leq n\|F\|_2 \sec(\theta_j).$$

Putting this bound on the columns of $\tilde{F}$ together with (1), we have the Bauer-Fike theorem.

Theorem 1 *Suppose $A \in \mathbb{C}^{n\times n}$ is diagonalizable with eigenvalues $\lambda_1, \ldots, \lambda_n$. Then all the eigenvalues of $A + F$ are in the region*

$$\bigcup_j \mathcal{B}_{n\|F\|_2 \sec(\theta_j)}(\lambda_j),$$

where θ_j is the acute angle between the row and column eigenvectors for λ_j, and any connected component $\mathcal{G}$ of this region that contains exactly m eigenvalues of A will also contain exactly m eigenvalues of $A + F$.