Midterm

You may use any standard reference (book, notes, HW solutions, MATLAB help), but please do not consult with classmates or others apart from me. You may use any MATLAB functions you wish in your solutions, and you *should* use MATLAB (or Octave) to check correctness of your codes. You will be graded on the correctness (and correct efficiency) of your solutions, but include derivations, test cases, etc. that you think will help me evaluate your solutions. Submit via CMS by Oct 22.

1. Let $A = I + uv^T$. Give an $O(n^2)$ time function to factor $A = LU$ (assume no pivoting is needed). Your code should have the form:

```
function [L,U] = p1lu(u, v)
%
% Compute L*U = I + u*v', where L is unit lower triangular
% and U is upper triangular.
```

Hint: The Schur complements retain the identity-plus-rank-one structure.

2. Let $A : \mathbb{R} \to \mathbb{R}^{n \times n}$ be a differentiable matrix valued function, and define

$$B(s) = \begin{bmatrix} A(s) & b \\ c^T & d \end{bmatrix}$$

where $b, c \in \mathbb{R}^n$ and $d \in \mathbb{R}$ are randomly chosen nonzero vectors. The scalar function $f(s) = e_{n+1}^T B(s)^{-1} e_{n+1}$ has a simple zero exactly when $A(s)$ is nonsingular with a one-dimensional null space. Given $A(s)$, $\dot{A}(s)$, b, c, and d, write a function `p2feval` that computes $f(s)$ and $f'(s)$. Your function should use at most one factorization (don't use `inv`!), though you may use any number of triangular solves. The code should have the interface

```
function f, df = p2feval(A, dA, b, c, d)
% Input:
%   A   = A(s)
%   dA  = d/ds A(s)
%   b,c = column vector of length n
%   d   = scalar value
% Output:
%   f   = f(s)
%   df  = d/ds f(s)
```

3. Let $M \in \mathbb{R}^{n \times n}$ be symmetric and positive definite, so that M defines an inner product $\langle x, y \rangle_M = y^T M x$ and an associated norm $\|x\|_M = \sqrt{\langle x, x, \rangle_M}$. Given $A \in \mathbb{R}^{n \times n}$, write a MATLAB function to compute the operator norm $\|A\|_M$ induced by this vector norm. Your code should look like

```
function normA = p3mnorm(A,M)
% Inputs:
%   A     = n-by-n real matrix
%   M     = n-by-n symmetric positive definite matrix
% Output:
%   normA = operator norm of A in the M inner product norm.
```

Hint: Use the Cholesky factorization of M.

4. Let $M \in \mathbb{R}^{n \times n}$ be symmetric and positive definite, so that M defines an inner product $\langle x, y \rangle_M = y^T M x$ and an associated norm $\|x\|_M = \sqrt{\langle x, x, \rangle_M}$. Implement a function to apply a generalized Householder reflector H for a given normal vector u that satisfies

1. $H^T M H = M$
2. $Hu = -u$
3. $Hv = v$ for all v s.t. $v^T M u = 0$.

Your code should look like

```
function y = p4applyH(Mu,u,x)
% Inputs:
%   u  = nonzero length n real vector
%   Mu = M*u
% Output:
%   y = H*x where H is the generalized reflector for u
```

This function should run in $O(n)$ time.

5. Suppose $A, B \in \mathbb{R}^{m \times n}$ where $m > n$ and A is full rank. Devise an $O(mn^2)$ algorithm to find the upper triangular $U \in \mathbb{R}^{n \times n}$ to minimize $\|AU - B\|_F^2$. Your code should look like

```
function [U] = p5solve(A,B)
%
% Find upper triangular U to minimize norm(A*U-B,'fro').
```