

HW 5

Due Nov 19.

1: Approximate inverses Suppose the eigenvalues of a symmetric matrix A all lie in an interval $[\alpha, \beta]$ that does not contain zero, and that p is some polynomial such that $|p(z) - 1/z| \leq C$ for any $z \in [\alpha, \beta]$. Argue that for any b ,

$$\|A^{-1}b - p(A)b\|_2 \leq C\|b\|_2.$$

2: Symmetric QR sweep Implement a routine to do one implicit iteration of shifted QR on a symmetric tridiagonal T ; that is,

- Compute an orthogonal G_1 so that

$$G_1 \begin{bmatrix} t_{11} - \sigma \\ t_{12} \end{bmatrix} = \begin{bmatrix} \alpha \\ 0 \end{bmatrix}$$

I recommend doing this with the `givens` command in MATLAB.

- Apply G_1 to the first two rows of T , and apply G_1^T to the first two columns. This will create a “bulge” from the tridiagonal structure in the $(1, 3)$ and $(3, 1)$ entries.
- Compute G_2, G_3, \dots, G_{n-1} in order to move the “bulge” to the $(2, 4)$ and $(4, 2)$ entries, then the $(3, 5)$ and $(5, 3)$ entries, and so on until the bulge is chased all the way off the end and the matrix is restored to tridiagonal form.

Your function should take the form

```
function T = p2sweep(T, sigma)
```

Ideally, one sweep should run in $O(n)$ time, though I recommend initially doing the $O(n^2)$ version — and you may want to look at the code in the lecture notes!

In addition to submitting the code, you should submit a semi-logarithmic plot of the $T_{n,n-1}^{(k)}$ entry of the iterates computed via the Wilkinson shift strategy starting from a random symmetric tridiagonal matrix, i.e.

```

Tk = T;
for k = 1:niter
    e(k) = Tk(end,end-1);
    Tk = p2sweep(Tk, Tk(end,end));
    if abs(e(k)) < 1e-15, break; end
end
semilogy(abs(e));

```

Do you observe cubic convergence?

3: Eigenvector perturbations Suppose $A = Q\Lambda Q^T$ is a symmetric matrix with all eigenvalues distinct. Let E be another symmetric matrix. For small enough t , we can write a smooth eigenvalue decomposition

$$A + tE = U(t)L(t)U(t)^T$$

where $U(0) = Q$, $L(0) = \Lambda$, and $U(t)$ and $L(t)$ are respectively orthogonal and diagonal for all sufficiently small t . Write a routine to compute $\dot{U}(0)$ and $\dot{L}(0)$. Your routine should have the form

```
function [Udot,Ldot] = p3deriv(Q0,L0,E)
```

What do you notice about the relationship between closeness of the eigenvalues and the rate at which the eigenvectors change?

Hint: Recall that $Q^T\dot{Q} = S$ is skew symmetric.