HW 3

Due Oct 10.

1: Tree climbing Let p be a *parent vector* for a tree with nodes $1, \ldots, n$, i.e. p_i is the index of the parent of node i (or zero if i is the root). Assume that $p_i < i$, so that children appear after parents in the ordering. Now, suppose that A is an $n \times n$ matrix such that

$$A_{ij} = \begin{cases} \alpha_i, & i = j, \\ \beta_j, & i = p_j, \\ \beta_i, & j = p_i, \\ 0, & \text{otherwise}, \end{cases}$$

and that A is positive definite. Write a function (not using the sparse intrinsics in MATLAB) that solves linear systems of the form $Ax = b$ in $O(n)$ time. Your function should look like

```
function [x] = p1tree(p, alpha, beta, b);
% Equivalent to
%   A = diag(alpha);
%   for k = 2:n
%     A(k,p(k)) = beta(k);
%     A(p(k),k) = beta(k);
%   end
%   x = A\b;
```

Alternately, you may implement an equivalent Python routine with an equivalent input/output signature.

2: Modified metrics Let $M \in \mathbb{R}^{n \times n}$ be symmetric and positive definite, and let $\langle x, y \rangle_M \equiv y^T M x$ and $\|x\|_M = \sqrt{\langle x, x \rangle_M}$ be the corresponding induced norm and inner product. $W \in \mathbb{R}^{n \times n}$ is M-orthogonal if $W^T M W = M$.

1. Write the normal equations for minimizing $\|Ax - b\|_M^2$.
2. Given $A \in \mathbb{R}^{n \times n}$, write a function to compute $A = WR$, where W is M-orthogonal and R is upper triangular. Your function should take the form

   ```
   function [W,R] = p2wr(A,M)
   ```

3: An extended system Let $A \in \mathbb{R}^{m \times n}$, $m \geq n$, be full rank.

1. Show that if
$$\begin{bmatrix} I & A \\ A^T & 0 \end{bmatrix} \begin{bmatrix} r \\ x \end{bmatrix} = \begin{bmatrix} b \\ 0 \end{bmatrix},$$
then x minimizes $\|Ax - b\|_2$.

2. What is the two-norm condition number of the coefficient matrix in part 1 in terms of the singular values of A?

3. Give an explicit expression for the inverse of the coefficient matrix, as a block 2-by-2 matrix.

4: Continuous connections Find the sixth-degree polynomial $p(x)$ that best approximates $\cos(x)$ on $[-\pi, \pi]$ in a least squares sense; that is, minimize

$$R(p) = \int_{-\pi}^{\pi} (p(x) - \cos(x))^2 \, dx.$$

Note: For k even, the integrals

$$b_k = \int_{-\pi}^{\pi} x^k \cos(x) \, dx,$$

satisfy the recurrence $b_0 = 0$ and $b_k = -k\left(2\pi^{k-1} + (k-1)b_{k-2}\right)$.