HW 2

Due: Monday, Sep 24.

Remember that you may (and should!) talk about the problems amongst yourselves, or discuss them with me, providing attribution for any good ideas you might get – but your final write-up should be your own.

1: Inverting a structured matrix Let $L = \text{tril}(uv^T)$ where $u \in \mathbb{R}^n$ and $v \in \mathbb{R}^n$ are componentwise nonzero.

1. Write an $O(n)$ code to solve $Lx = b$ by forward substitution.
2. Argue that L^{-1} is a lower bidiagonal matrix.

2: Deriving Sherman-Morrison-Woodbury Suppose $A = B + uv^T$ where $B \in \mathbb{R}^{n \times n}$, $u, v \in \mathbb{R}^n$, and we assume A and B are nonsingular.

1. Show that if x satisfies $Ax = b$, then there is some y such that
$$\begin{bmatrix} B & u \\ v^T & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} b \\ 0 \end{bmatrix}.$$
2. Using Schur complements, find a formula for y. Use this in turn to derive a formula for A^{-1} in terms of B, u, and v.

3: The flexible residual Suppose $\hat{x}$ is an approximate solution to a linear system, and define the *residual vector* $r = b - A\hat{x}$.

1. Find a rank-one matrix E such that $(A + E)\hat{x} = b$ where $\|E\|_2 = \|r\|_2/\|\hat{x}\|_2$.
2. Show the relative error bound
$$\frac{\|\hat{x} - x\|_\infty}{\|\hat{x}\|_\infty} \leq \frac{\| \, |A^{-1}| \, |r| \, \|_\infty}{\|\hat{x}\|_\infty}$$
Here $|A^{-1}|$ and $|r|$ indicate elementwise absolute values.
3. Show that if $R = \text{diag}(r)$, then
$$\| \, |A^{-1}| \, |r| \, \|_\infty = \|A^{-1}R\|_\infty$$
This can be estimated efficiently using Hager's method; consequently, we can actually estimate the right hand side of the bound in (2) efficiently. This is done in the LAPACK routines `sgesvx` and `dgesvx`.

4: Resistance is futile The *conductance matrix* G for a network of resistors is a (typically sparse) matrix in which the (i, j) off-diagonal entry is $-1/R_{ij}$ if there is a resistor with resistance of R_{ij} connecting node i and node j, and zero if there is no direct connection between the two nodes. The diagonal entries are given by

$$g_{ii} = -\sum_{j \neq i} g_{ij}.$$

Assuming the network is connected, the *effective resistance* R between node 1 and n can be computed as follows. Let $G_{\text{red}} = G(1 : n-1, 1 : n-1)$ be the conductivity matrix with the last row and column removed. If the network is connected, then G_{red} will be nonsingular, and the $(1, 1)$ entry of G_{red}^{-1} is the effective resistance.

1. Describe how to compute the effective resistance using one linear solve involving G_{red}.
2. Assuming a resistor connects i to j, write an expression for $\partial R/\partial R_{ij}$.

   Hint: Recall that if A is invertible,

   $$\left.\frac{d}{ds}\right|_{s=0} (A + sE)^{-1} = -A^{-1}EA^{-1}$$

3. Describe how to compute $\partial R/\partial R_{ij}$ for *all* resistors in the network by solving one linear system with G_{red} followed by a constant amount of work per resistor.