

Lecture 17: Iterative Methods and Sparse Linear Algebra

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Logistics

- ▶ HW 3 extended to Wednesday after break
- ▶ HW 4 should come out Monday after break
- ▶ Still need project description and group by Friday

Reminder: World of Linear Algebra

- ▶ Dense methods
 - ▶ Direct representation of matrices with simple data structures (no need for indexing data structure)
 - ▶ Mostly $O(n^3)$ factorization algorithms
- ▶ Sparse direct methods
 - ▶ Direct representation, keep only the nonzeros
 - ▶ Factorization costs depend on problem structure (1D cheap; 2D reasonable; 3D gets expensive; not easy to give a general rule, and NP hard to order for optimal sparsity)
 - ▶ Robust, but hard to scale to large 3D problems
- ▶ Iterative methods
 - ▶ Only *need* $y = Ax$ (maybe $y = A^T x$)
 - ▶ Produce successively better (?) approximations
 - ▶ Good convergence depends on *preconditioning*
 - ▶ Best preconditioners are often hard to parallelize

Linear Algebra Software: MATLAB

```
% Dense (LAPACK)
```

```
[L,U] = lu(A);
```

```
x = U \ (L \ b);
```

```
% Sparse direct (UMFPACK + COLAMD)
```

```
[L,U,P,Q] = lu(A);
```

```
x = Q * (U \ (L \ (P * b)));
```

```
% Sparse iterative (PCG + incomplete Cholesky)
```

```
tol = 1e-6;
```

```
maxit = 500;
```

```
R = cholinc(A, '0');
```

```
x = pcg(A, b, tol, maxit, R', R);
```

Linear Algebra Software: the Wider World

- ▶ Dense: LAPACK, ScaLAPACK, PLAPACK
- ▶ Sparse direct: UMFPACK, TAUCS, SuperLU, MUMPS, Pardiso, SPOOLES, ...
- ▶ Sparse iterative: too many!
- ▶ Sparse mega-libraries
 - ▶ PETSc (Argonne, object-oriented C)
 - ▶ Trilinos (Sandia, C++)
- ▶ Good references:
 - ▶ *Templates for the Solution of Linear Systems* (on Netlib)
 - ▶ Survey on “Parallel Linear Algebra Software” (Eijkhout, Langou, Dongarra – look on Netlib)
 - ▶ ACTS collection at NERSC

Software Strategies: Dense Case

Assuming you want to *use* (vs develop) dense LA code:

- ▶ Learn enough to identify right algorithm (e.g. is it symmetric? definite? banded? etc)
- ▶ Learn high-level organizational ideas
- ▶ Make sure you have a good BLAS
- ▶ Call LAPACK/ScaLAPACK!
- ▶ For n large: wait a while

Software Strategies: Sparse Direct Case

Assuming you want to use (vs develop) sparse LA code

- ▶ Identify right algorithm (mainly Cholesky vs LU)
- ▶ Get a good solver (often from list)
 - ▶ You *don't* want to roll your own!
- ▶ *Order your unknowns* for sparsity
 - ▶ Again, good to use someone else's software!
- ▶ For n large, 3D: get lots of memory and wait

Software Strategies: Sparse Iterative Case

Assuming you want to use (vs develop) sparse LA software...

- ▶ Identify a good algorithm (GMRES? CG?)
- ▶ Pick a good preconditioner
 - ▶ Often helps to know the application
 - ▶ ... *and* to know how the solvers work!
- ▶ Play with parameters, preconditioner variants, etc...
- ▶ Swear until you get acceptable convergence?
- ▶ Repeat for the next variation on the problem

Frameworks (e.g. PETSc or Trilinos) speed experimentation.

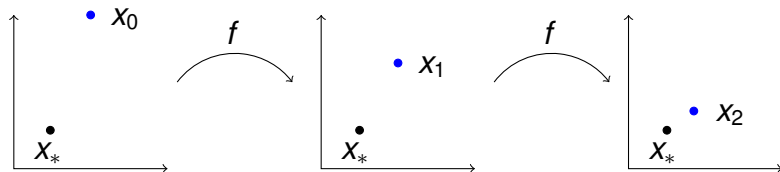
Software Strategies: Stacking Solvers

(Typical) example from a bone modeling package:

- ▶ Outer load stepping loop
- ▶ Newton method corrector for each load step
- ▶ Preconditioned CG for linear system
- ▶ Multigrid preconditioner
- ▶ Sparse direct solver for coarse-grid solve (UMFPACK)
- ▶ LAPACK/BLAS under that

First three are high level — I used a scripting language (Lua).

Iterative Idea



- ▶ f is a *contraction* if $\|f(x) - f(y)\| < \|x - y\|$.
- ▶ f has a unique *fixed point* $x_* = f(x_*)$.
- ▶ For $x_{k+1} = f(x_k)$, $x_k \rightarrow x_*$.
- ▶ If $\|f(x) - f(y)\| < \alpha \|x - y\|$, $\alpha < 1$, for all x, y , then

$$\|x_k - x_*\| < \alpha^k \|x - x_*\|$$

- ▶ Looks good *if* α not too near 1...

Stationary Iterations

Write $Ax = b$ as $A = M - K$; get fixed point of

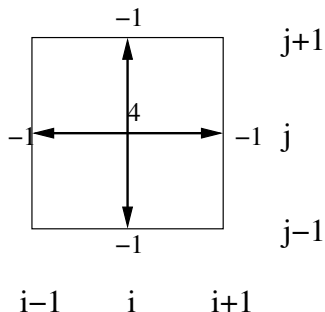
$$Mx_{k+1} = Kx_k + b$$

or

$$x_{k+1} = (M^{-1}K)x_k + M^{-1}b.$$

- ▶ Convergence if $\rho(M^{-1}K) < 1$
- ▶ Best case for convergence: $M = A$
- ▶ Cheapest case: $M = I$
- ▶ Realistic: choose something between
 - Jacobi $M = \text{diag}(A)$
 - Gauss-Seidel $M = \text{tril}(A)$

Reminder: Discretized 2D Poisson Problem



$$(Lu)_{i,j} = h^{-2} (4u_{i,j} - u_{i-1,j} - u_{i+1,j} - u_{i,j-1} - u_{i,j+1})$$

Jacobi on 2D Poisson

Assuming homogeneous Dirichlet boundary conditions

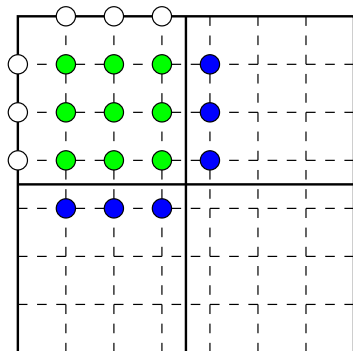
```
for step = 1:nsteps

    for i = 2:n-1
        for j = 2:n-1
            u_next(i,j) = ...
                ( u(i,j+1) + u(i,j-1) + ...
                  u(i-1,j) + u(i+1,j) )/4 - ...
                h^2*f(i,j)/4;
        end
    end
    u = u_next;

end
```

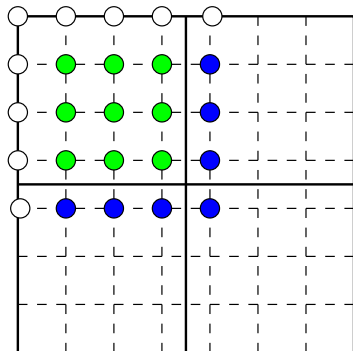
Basically do some averaging at each step.

Parallel version (5 point stencil)



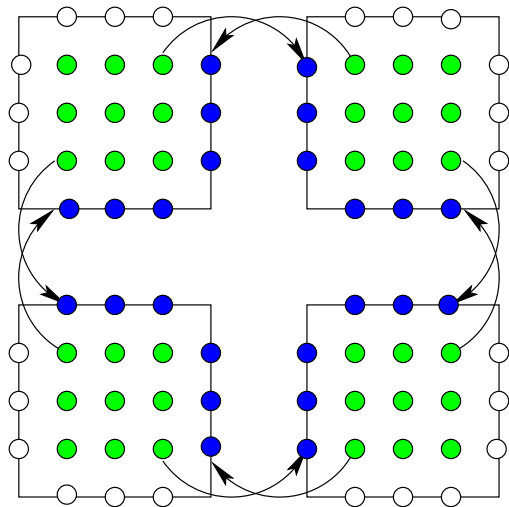
Boundary values: white
Data on P0: green
Ghost cell data: blue

Parallel version (9 point stencil)



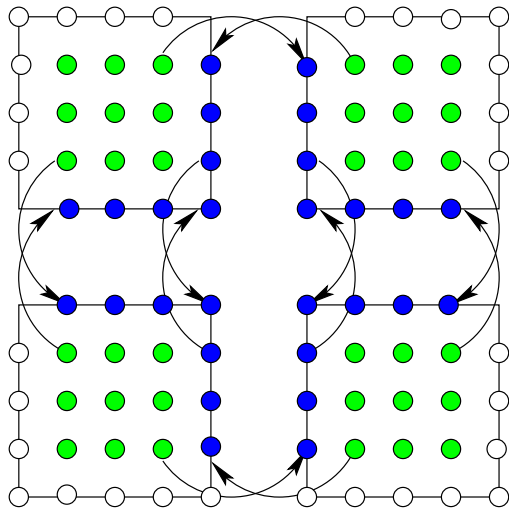
Boundary values: white
Data on P0: green
Ghost cell data: blue

Parallel version (5 point stencil)



Communicate ghost cells before each step.

Parallel version (9 point stencil)



Communicate in two phases (EW, NS) to get corners.

Gauss-Seidel on 2D Poisson

```
for step = 1:nsteps

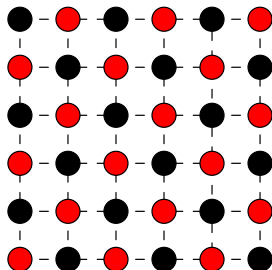
    for i = 2:n-1
        for j = 2:n-1
            u(i,j) = ...
                ( u(i,j+1) + u(i,j-1) + ...
                  u(i-1,j) + u(i+1,j) )/4 - ...
                h^2*f(i,j)/4;
        end
    end

end

end
```

Bottom values depend on top; how to parallelize?

Red-Black Gauss-Seidel



Red depends only on black, and vice-versa.
Generalization: multi-color orderings

Red black Gauss-Seidel step

```
for i = 2:n-1
  for j = 2:n-1
    if mod(i+j,2) == 0
      u(i,j) = ...
    end
  end
end
```

```
for i = 2:n-1
  for j = 2:n-1
    if mod(i+j,2) == 1,
      u(i,j) = ...
    end
  end
end
```

Parallel red-black Gauss-Seidel sketch

At each step

- ▶ Send black ghost cells
- ▶ Update red cells
- ▶ Send red ghost cells
- ▶ Update black ghost cells

More Sophistication

- ▶ Successive over-relaxation (SOR): extrapolate Gauss-Seidel direction
- ▶ Block Jacobi: let M be a block diagonal matrix from A
 - ▶ Other block variants similar
- ▶ Alternating Direction Implicit (ADI): alternately solve on vertical lines and horizontal lines
- ▶ Multigrid

These are mostly just the opening act for...

Krylov Subspace Methods

What if we only know how to multiply by A ?
About all you can do is keep multiplying!

$$\mathcal{K}_k(A, b) = \text{span} \{ b, Ab, A^2b, \dots, A^{k-1}b \}.$$

Gives surprisingly useful information!

Example: Conjugate Gradients

If A is symmetric and positive definite, $Ax = b$ solves a minimization:

$$\phi(x) = \frac{1}{2}x^T Ax - x^T b$$
$$\nabla\phi(x) = Ax - b.$$

Idea: Minimize $\phi(x)$ over $\mathcal{K}_k(A, b)$.

Basis for the *method of conjugate gradients*

Example: GMRES

Idea: Minimize $\|Ax - b\|^2$ over $\mathcal{K}_k(A, b)$.

Yields *Generalized Minimum RESidual* (GMRES) method.

Convergence of Krylov Subspace Methods

- ▶ KSPs are *not* stationary (no constant fixed-point iteration)
- ▶ Convergence is surprisingly subtle!
- ▶ CG convergence upper bound via *condition number*
 - ▶ Large condition number iff form $\phi(x)$ has long narrow bowl
 - ▶ Usually happens for Poisson and related problems
- ▶ *Preconditioned* problem $M^{-1}Ax = M^{-1}b$ converges faster?
- ▶ Whence M ?
 - ▶ From a stationary method?
 - ▶ From a simpler/coarser discretization?
 - ▶ From approximate factorization?

PCG

Compute $r^{(0)} = b - Ax$

for $i = 1, 2, \dots$

 solve $Mz^{(i-1)} = r^{(i-1)}$

$\rho_{i-1} = (r^{(i-1)})^T z^{(i-1)}$

 if $i == 1$

$p^{(1)} = z^{(0)}$

 else

$\beta_{i-1} = \rho_{i-1} / \rho_{i-2}$

$p^{(i)} = z^{(i-1)} + \beta_{i-1} p^{(i-1)}$

 endif

$q^{(i)} = Ap^{(i)}$

$\alpha_i = \rho_{i-1} / (p^{(i)})^T q^{(i)}$

$x^{(i)} = x^{(i-1)} + \alpha_i p^{(i)}$

$r^{(i)} = r^{(i-1)} - \alpha_i q^{(i)}$

end

Parallel work:

- ▶ Solve with M
- ▶ Product with A
- ▶ Dot products
- ▶ Axpys

Overlap comm/comp.

PCG bottlenecks

Key: fast solve with M , product with A

- ▶ Some preconditioners parallelize better!
(Jacobi vs Gauss-Seidel)
- ▶ Balance speed with performance.
 - ▶ Speed for set up of M ?
 - ▶ Speed to apply M after setup?
- ▶ Cheaper to do two multiplies/solves at once...
 - ▶ Can't exploit in obvious way — lose stability
 - ▶ Variants allow multiple products — Hoemmen's thesis
- ▶ Lots of fiddling possible with M ; what about matvec with A ?