Practice Midterm

This is around the right scale and form for a midterm for 4220, but it may be somewhat harder or easier than the actual midterm. The actual exam will be open book and notes, but limited to 50 minutes. In evaluating yourself, you may want to try the exam under those conditions.

1: True/False

1. Suppose $Ax = b$ and $(A + E)\hat{x} = b$. Then $\|\hat{x} - x\| \leq \kappa(A)\|E\|$.
2. If a and b are normalized floating point numbers and $a+b$ is in the range of normalized floating point numbers, then $\mathrm{fl}(a + b) = (a + b)(1 + \delta)$ where $|\delta| \leq \epsilon_{\mathrm{mach}}$.
3. Newton's iteration is quadratically convergent for $f(x) = x^2 = 0$ for starting points sufficiently near zero.
4. If A is singular, Gaussian elimination cannot compute $PA = LU$.
5. In Gaussian elimination with partial pivoting, all elements of L below the main diagonal have magnitude at most one.

Answer:

1. **False**. The condition number relates relative errors, not absolute errors.
2. **True**.
3. **False**. In this case, Newton's iteration is $x_{k+1} = x_k/2$, which converges linearly. We saw something like this in a homework.
4. **False**. Singular matrices can have LU factorizations; consider

$$L = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

5. **True**. This is the point of partial pivoting.

2: Fixed point fandango Consider the iteration

$$x_{k+1} = 10 - \exp(x_k).$$

The iteration has a fixed point $x_* \approx 2.0706$. For x_0 close enough to x_*, does the iteration converge? Explain by writing an error recurrence.

Answer: The iteration equation is

$$e_{k+1} = -(\exp(x_* + e_k) - \exp(x_*)) = -\exp(x_*)e_k + O(e_k^2)$$

Because $|\exp(x_*)| > 1$, the iteration diverges in general.

3: Norm! The *Frobenius norm* of a matrix A is

$$\|A\|_F = \sqrt{\sum_{i,j} a_{ij}^2}.$$

Show

1. The Frobenius norm is not an operator norm (hint: consider $\|I\|_F$).
2. The Frobenius norm is consistent with the two norm, i.e.

   $$\|Av\|_2 \leq \|A\|_F \|v\|_2.$$

   Hint: The Cauchy-Schwarz inequality states $|x \cdot y| \leq \|x\|_2 \, \|y\|_2$.

Answer:

1. For an operator norm,

   $$\|I\| = \max_{x \neq 0} \|x\|/\|x\| = 1.$$

   For the Frobenius norm, $\|I\|_F = \sqrt{n}$.

2. By Cauchy-Schwarz,

   $$(Av)_i^2 \leq \|A(i,:)\|_2^2 \|v\|_2^2$$

   Summing over i, we have

   $$\|Av\|_2^2 \leq \|A\|_F^2 \|v\|_2^2$$

4: Pseudoinverse Suppose $A \in \mathbb{R}^{n \times m}$ has full column rank, $n > m$.

1. Write the pseudoinverse in terms of A, the economy QR factorization of A, and the economy SVD of A.
2. Show that if $n = m$, then the pseudoinverse is the same as the inverse.
3. Give a *brief* geometric characterization of the null space of $A^\dagger$.

Answer:

1. $A^\dagger = (A^T A)^{-1} A^T = R^{-1} Q^T = V \Sigma^{-1} U^T$
2. $A^\dagger = (A^T A)^{-1} A^T = A^{-1} A^{-T} A^T = A^{-1}$ when A (and hence A^T) are square and invertible.
3. The null space of $A^\dagger$ is those vectors orthogonal to the range space of A.

5: Elimination and low rank Consider the matrix

$$A = I + uv^T$$

where u and v are elementwise positive, $\|u\|_1 < 1$ and $\|v\|_1 < 1$.

1. A must be diagonally dominant. Briefly state why.
2. Show that after one step of Gaussian elimination, the Schur complement has the form
$$S = I + \alpha u_2 v_2^T.$$
Write a simple expression for the coefficient α.

Answer:

1. Each column is $e_j + uv_j$. The jth element is $1 + u_j > 1$; the sum of the remaining element magnitudes is bounded by $\sum_i |u_i v_j| = \|u\|_1 |v_j| < 1$.
2. One step of Gaussian elimination gives
$$S = I + \frac{u_1 v_1}{1 + u_1 v_1} u_2 v_2^T$$