1 Chain Formulas and Theoretical Underpinning

Theorem 1. It takes RWF $O(n^2)$ flips to find a solution on a 2chain with $n$ variables.

Proof. Consider any assignment of the $n$ variables. A patch in the assignment is defined as a maximal set of consecutive variables that are assigned to the same value. For example, assignment 1001100011 has four patches, namely $x_2 = x_3 = 1$, $x_4 = x_5 = 0$, $x_6 = x_7 = x_8 = 0$, $x_9 = x_{10} = x_1 = 1$.

An important observation is that on a 2chain formula, a freebie flip will reduce the number of patches in the assignment by 2. This is because a freebie occurs when a variable is assigned to a different value from its two neighbors, and a clause containing this variable is chosen by RWF algorithm as the unsatisfied clause in consideration. The freebie flip will change the value of this variable, and therefore remove the patch composed of this single variable, and also merge its two neighbor patches into one patch.

Non-freebie flips (RW flips) neither reduce or increase the number of patches in the current assignment. Since an unsatisfied clause corresponds to a transition from a “1” patch to a “0” patch, these flips can be seen as moving the boundaries of the patches. Because if any variable in the unsatisfied clause in consideration constituted a patch of size 1, a freebie flip would have been triggered. Given that all relevant patches are of size 2 or more, and all flips happen on the boundaries of the patches, no patch can be eliminated or generated by RW flips.

On average, a random assignment falsifies $n/4$ clauses in a 2chain formula of $n$ variables. Therefore, there are approximately $n/2$ patches in the initial assignment, which means there are $n/4$ freebie flips in the RWF process. Note that an average initial assignment assigns $n/2$ variables to 1, and the other $n/2$ to 0. Therefore, even if the $n/4$ freebies flips all move the assignment to the same direction, there are still $n/4$ variables that need to be flipped in order to reach a satisfying assignment. These flips have to be done in the RW process, which takes $O(n^2)$ flips.

Theorem 2. It takes WalkSat $O(n^2)$ flips to find a solution on a 2chain with $n$ variables.

Proof. Consider a non-freebie greedy step of WalkSat on a 2chain formula. We claim this flip is equivalent to a RW flip. Since WalkSat makes a greedy move, there is no freebie variable exists in the unsatisfied clause in consideration. Now we consider the break value of each variable in this clause. Since no freebie exists, both break values are greater than 0. On the other hand, both break values cannot be greater than 1 because each variable only appears in one other clause of the formula. So we know they are both 1, and the probability of flipping each variable is 0.5. This is equivalent to making a RW move, and therefore, the behavior of WalkSat on 2chain is exactly the same as that of RWF.

In the discussion of cyclic chain formulas, we consider $x_n$ and $x_1$ as contiguous variables.