Theory Qualifier
January 16, 2006

Instructions

Please read these instructions and all questions carefully before proceeding.

There are five questions in all. Please answer three of them. Answer at least one of questions 1 and 2, at least one of questions 4 and 5, and a third question of your choice. Indicate clearly which three you are attempting. Do not attempt to answer more than three.

You have $2\frac{1}{2}$ hours. The exam is closed-book. All questions are of equal weight. All algorithms should be accompanied by a proof of correctness and complexity analysis unless otherwise directed.

The questions will be graded according to the following criteria in order of importance:

1. correctness and completeness
2. clarity, precision, and conciseness
3. optimality of result

Partial credit will be awarded where appropriate, so be sure to show all work. Unclear or irrelevant arguments will be penalized.

Good luck!
1. Let \( \#a(x) \) denote the number of occurrences of the symbol \( a \) in the string \( x \). Say whether the set

\[
(a + b)^*(aaa + bbb)(a + b)^* \cup \{x \in (a + b)^* \mid \#a(x) = \#b(x)\}
\]

is regular or nonregular. Give proof.

2. Show that if \( P = NP \), then there is a total recursive function that, given a description of a nondeterministic polynomial-time machine with an explicit time bound of precisely \( n^k \), produces a description of an equivalent deterministic polynomial time machine.

3. If \( G = (V, E) \) is a directed graph, \( S \) a subset of \( V \), and \( P \) a path in \( G \), we say that \( P \) dominates \( S \) if for every node \( w \in S \), there exists a node \( v \in P \) such that \((v, w)\) is an edge of \( G \). Prove that it is \( NP \)-complete to decide for a given \( G = (V, E) \) and \( S \subseteq V \) whether \( G \) contains a simple path \( P \) that dominates \( S \).

4. Say we have \( m \) machines \( M_1, \ldots, M_m \) that can process jobs of \( n \) different types. Each job requires one unit of time on one machine, and no machine can process more than one job at a time. For each job type \( j \), there is a designated subset of machines \( S_j \subseteq \{M_1, \ldots, M_m\} \) that can process jobs of type \( j \). Any job of type \( j \) can be assigned to any machine in \( S_j \). The sets \( S_j \) are not necessarily disjoint.

Suppose we are given a list \( k_1, \ldots, k_n \) of the number of jobs of each type. Starting at time 0, we wish to process all the jobs, minimizing the time by which all jobs are completed. Give a polynomial-time algorithm to find an optimal schedule. (Hint. You might first want to give an algorithm that, for a given target time \( t \), decides if there is a schedule that completes all jobs by time \( t \).)
5. Let $G = (V, E)$ be a directed graph with no multiple edges or loops. A *feedback edge set* of $G$ is a set of edges $F \subseteq E$ such that removing $F$ leaves the graph acyclic. A useful way of thinking about feedback edge sets is in terms of total orderings of the vertices. If $<$ is a total ordering of the vertices, then the set $\{(u, v) \in E \mid v < u\}$ (that is, the set of edges in $E$ that go from a larger to a smaller vertex in the ordering) is a feedback edge set. Conversely, given any *minimal* feedback edge set $F$, there is a total ordering $<$ such that $F = \{(u, v) \in E \mid v < u\}$.

A directed graph $G = (V, E)$ is said to be a *tournament* if for every pair of distinct vertices $u, v$, either $(u, v) \in E$ or $(v, u) \in E$ but not both. In a random tournament, the probability that $(u, v) \in E$ is equal to the probability that $(v, u) \in E$; that is, both events occur with probability $1/2$.

(a) Prove that a random tournament has a feedback edge set with expected number of edges at most $n^2/4$.

(b) Give a deterministic $O(n^2)$ time algorithm for finding a feedback edge set of at most $n^2/4$ edges in an arbitrary tournament.

END OF EXAM