CS 421: Numerical Analysis
Fall 2000
Final Exam

Handed out: Friday, Dec. 8.

This exam has 12 questions and 120 points total on 3 pages (including this page). You have 120 minutes to complete all the questions. Write your answers in the booklet. This exam is closed-book and closed-note, but you may consult a prepared sheet of notes (one page, $8\frac{1}{2}'' \times 11''$ written on both sides).

If you are a Computer Science PhD student who is taking this exam as the NA Q-exam, write your code number on the booklet instead of your name.

Everyone else: please write your name on the booklet.
The following questions are short answer—a single phrase or formula suffices.

1. [5 points] How many flops, accurate to the leading term, are required for back substitution to solve the upper triangular system $Ux = b$, $U \in \mathbb{R}^{n \times n}$ and $b \in \mathbb{R}^n$?

2. [5 points] Write down the formula for the 1-norm of a vector $x \in \mathbb{R}^n$.

3. [5 points] What is the relationship between $\text{cond}(A)$ and $\text{cond}(2A)$, where “cond” denotes the 2-norm condition number and $A$ is an $n \times n$ nonsingular matrix?

4. [5 points] If the eigenvalues of $A \in \mathbb{R}^{3 \times 3}$ are 6, $-2$, 0, then what are the eigenvalues of $(I - A)^{-1}$?

5. [5 points] Given $A \in \mathbb{R}^{m \times n}$, $m \geq n$, and $b \in \mathbb{R}^m$, a standard algorithm for minimizing $\|Ax - b\|_2$ is QR-factorization followed by back-substitution. This algorithm fails if $\text{rank}(A) < n$. Specifically, what goes wrong?

6. [5 points] Let $Q$ be an $n \times n$ orthogonal matrix. What are its singular values?

7. [5 points] Name a method that could be used to solve for the next iterate (i.e., for $y^{(k+1)}$) of the Backward Euler method.

8. [5 points] The state of Florida has been in the news a lot during the past month. How many letters are in the word Florida? It’s OK to count and then recount.
These questions require longer answers.

9. [20 points] Suppose $A \in \mathbb{R}^{n \times n}$ is a square nonsingular matrix. Suppose one is given $b \in \mathbb{R}^n$ and entries $2, \ldots, n$ of a vector $x \in \mathbb{R}^n$. Consider the problem of choosing $x(1)$ to minimize $\|Ax - b\|_2$ given all the other data. Computing $x(1)$ is a linear least-squares problem. Write down the normal equations of this linear least-squares problem.

10. [20 points] Construct a polynomial function $p(x)$ such that Newton’s method for finding the roots $p(x)$ starting at $x^{(0)} = 0$ fails to converge, but rather generates the sequence $x^{(0)} = 0, x^{(1)} = 1, x^{(2)} = 0, x^{(3)} = 1$, etc. [Hint: there is a degree-2 solution.]

11. [20 points] Let $G \in \mathbb{R}^{2 \times 2}$ be the Givens rotation $[\cos \theta, \sin \theta; -\sin \theta, \cos \theta]$. Assume that $\theta$ is not a multiple of $\pi$. Argue that $G$ cannot have any real eigenvalues. How well would you expect the (unshifted) power method for finding eigenvectors of $G$, initialized with a real vector, to work?

12. [20 points] Consider a scalar initial value problem of the form $dy/dt = f(y)$ where $y(0) = y_0$. Suppose $y_0 > 0$, and suppose $f$ is a decreasing, continuously-differentiable function such that $f(0) = 0$. (So, in particular, $f(z) < 0$ for $z > 0$ and $f(z) > 0$ for $z < 0$.) It is a known theorem that the true solution to this IVP is a positive-valued function. If the Euler method is applied to this IVP, is the computed solution guaranteed to be positive-valued? How about the Backward Euler method? Explain your answers.