1 A Facility Location Game (A. Vetta 2002)

1.1 Motivation

The facility location game is related to the facility location problem. The facility location problem is a topology problem. Given a topology one wishes to place facilities. Each facility exists to serve clients, and the problem ask how to minimise both client distance and facility creation. This problem has many applications to the real world. However it presumes that there is only one possible entity that can place facilities. The facility location game simplifies the topology somewhat, but allows multiple players to place facilities.

1.2 The Elements of the Game

The game consists of markets, suppliers and locations. There are \( m \) markets, each denoted \( m_i \). There are \( k \) suppliers, and each supplier \( k \) has an associated set of locations \( L^k \subseteq L \), where \( L \) is the set of all locations and \( L_j \) is a particular location.

Each market \( m_i \) has a value \( \Pi_i \) associated with it. This is the value it receives when served. Alternately one can think of it as the highest value that it is willing to pay the suppliers for there supplies. Between each market \( m_i \) and each location \( L_j \) there is an edge with weight \( \lambda_{ij} \). This represents the cost of serving supplies to \( m_i \) from \( L_j \). Each supplier \( k \) may build a single facility at a location \( L_i \) contained within \( L^k \). Because only the suppliers have a direct influence on their choice of facility, it is the suppliers who play the game. A solution is then just a mapping of suppliers to locations. Having picked a location, a supplier then tries to maximise his profit from that location.

We note that a supplier will only supply a market if he can profit from it. So, anywhere where
\( \lambda_{ij} \geq \Pi_i \) the supplier will not attempt to serve the market. This means that \( \forall \lambda_{ij} \geq \Pi_i \) we can set \( \lambda_{ij} = \Pi_i \). We will do so. Thus, \( \lambda_{ij} \) must satisfy: \( 0 \leq \lambda_{ij} \leq \Pi_i \).

### 1.3 Characterising the Solution to this Game

Given a solution we would like some way to characterise the quality of this solution. We note that once we have a solution, we can determine what the suppliers will do. To do this we look at a market \( m_i \). In the solution there are \( L^i \subseteq L \) open facilities that can serve \( m_i \). Clearly the supplier that will end up serving \( m_i \) is the supplier that can serve \( m_i \) at cheapest cost, as he can undercut all of his competitor. So we say that \( m_i \) is served by \( \sigma(i) \), where we define \( \sigma(i) \):

\[
\sigma(i) = \arg\min_{j \in L^i} \lambda_{ij}
\]

The price that this supplier can charge should be equal to the cost that his closest competitor incurs. If he sets it lower, then there is still profit to be made, by raising the price to that level. However if he sets it higher, than his closest competitor can undercut him and he will get no business. So, we say that \( m_i \) pays a price \( p_i \) where we define \( p_i \):

\[
p_i = \min_{j \in L^i, j \neq \sigma(i)} \lambda_{ij}
\]

Note that \( \sigma(i) \) defines a location, while \( p_i \) defines a cost.

### 1.4 Quality of the Solution to this Game

To measure the quality of a solution we try and see who benefits when \( m_i \) is served at a price \( p_i \). Now, \( m_i \) is willing to pay up to \( \Pi_i \), but it benefits more by paying less. So, a natural benefit function for \( m_i \) is \( \Pi_i - p_i \). At the same time, the supplier at \( \sigma(i) \) is getting \( p_i \) but paying \( \lambda_{i\sigma(i)} \) so the natural benefit function for \( \sigma(i) \) is \( p_i - \lambda_{i\sigma(i)} \). The total benefit is just the sum of the benefits of the markets, and the benefits of the suppliers. But we note that at most one supplier is getting any benefit from a given market, which enables us to write.

\[
TotalBenefit = \sum_{i \mid m_i \text{ served}} (\Pi_i - p_i) + \sum_{i \mid m_i \text{ served}} (p_i - \lambda_{i\sigma(i)})
\]

\[
TotalBenefit = \sum_{i \mid m_i \text{ served}} (\Pi_i - \lambda_{i\sigma(i)})
\]

To make this slightly cleaner we will assume that if \( m_i \) is not served by any supplier, it is in fact served by some supplier at cost. That is, \( \Pi_i = \lambda_{i\sigma(i)} \). This does not change the value of the total benefit, but it gives us the slightly cleaner form:

\[
TotalBenefit = \sum_{all \ i} (\Pi_i - \lambda_{i\sigma(i)})
\]
2 Nash Equilibria in the Facility Location Game

2.1 Pertinent Questions

There are three questions we typically ask about Nash Equilibria:

- Does a Nash equilibrium exist?
- If it does exist, is it unique?
- If it does exist, how does it compare to the optimal solution?

The answers to these questions follow.

2.2 The Existence of a Nash Equilibrium

**Theorem 1** The Facility Location game is a Potential Game with Potential function $\Phi$ where

$$\Phi = \sum_{all \ i} \lambda_{i\sigma(i)}$$

**Proof.** We need to show that $\Phi$ tracks a player’s benefit change when he switches. So, let us consider user $k$. Let us take him out of the game. That is, let him spontaneously decide to leave. Now we consider all of the $\lambda_{i\sigma(i)}$ where $\sigma(i) = k$. We note that by the definition of $p_i$ we get $\lambda_{i\sigma^{new}(i)} = p_i$. So

$$\Delta \Phi = \sum_{i|\sigma(i)=k} (p_i - \lambda_{i\sigma^{old}(i)})$$

This is exactly the loss of profits that user $k$ cost himself when he left the graph. So $\Phi$ tracks the losses properly.

Now we consider what happens when $k$ jumps back into the game. At each market he’ll either get nothing in which case $\lambda_{i\sigma(i)}$ won’t change, or (if $k = \sigma^{new}(i)$) he’ll get $p_i^{new} - \lambda_{i\sigma^{new}(i)}$ where $p_i^{new}$ is trivially $\lambda_{i\sigma^{old}(i)}$. But this is exactly the difference in $\Phi$ for that market. When we consider all of the markets that the user touches, we note that the benefit to the user is exactly the increase in $\Phi$. So $\Phi$ accurately tracks the change in each user’s benefit, so $\Phi$ is a potential function.

**Corollary 1** A Nash equilibrium exists,

**Proof.** In a potential game, the minima of the potential function ($\Phi$) are Nash equilibria, and we can clearly get to these minima by forcing users to change if and only if doing so would lessen $\Phi$ and forbidding them to do so when it wouldn’t.

**Corollary 2** The global minimum of $\Phi$ is the optimal solution. So the best Nash equilibrium is also the best solution.

**Proof.** Note that:

$$Total\ Benefit = \sum_{all \ i} (\Pi_i - \lambda_{i\sigma(i)})$$

But $\sum_{all \ i} \Pi_i$ is a constant. So then to maximise the total benefit we have to minimise $\sum_{all \ i} \lambda_{i\sigma(i)}$ which is exactly what the global minimum of $\Phi$ does.
2.3 The Uniqueness of a Nash Equilibrium

The following example, where $L^1 = \{1, 2\}$ and $L^2 = \{3, 4\}$ and $\forall i \Pi_i = 1$, shows that there is not a unique Nash Equilibrium.

![Graph showing two Nash Equilibria](image)

Figure 2: A Facility Location Game with two Nash Equilibria. Edges not shown on the graph have $\lambda_{ij} = 1$, otherwise they have $\lambda_{ij} = 0$.

We note that the red equilibrium has a total benefit of 4, while the green equilibrium has a total benefit of 2, or half that of the optimal. This leads nicely to the next section of the notes, namely:

2.4 The Quality of a Nash Equilibrium

**Theorem 2** The total benefit of any Nash Equilibrium is at least 1/2 of the total benefit of the optimal solution.

**Proof.** We want to compare a Nash equilibrium to an optimal solution. To do this we will use regular notation for the Nash equilibrium and primed notation for the optimal solution. As an example $\sigma(i)$ is the location $m_i$ gets assigned to in the Nash equilibrium, and $\sigma'(i)$ is the location $m_i$ gets assigned to in the optimal solution.

Now we introduce some more notation. Let $val(k)$ be the total profit that supplier $k$ gets in the Nash equilibrium, and let $val'(k)$ be the total profit that supplier $k$ gets if everyone else keeps their location in the Nash equilibrium, but supplier $k$ uses his location in the optimal solution. Clearly $val(k) \geq val'(k)$ as $val(k)$ refers to a Nash equilibrium. Finally we define $\delta(i)$ as

$$
\delta(i) = \lambda_{i\sigma(i)} - \lambda_{i\sigma'(i)}
$$

This gives use the following Lemma:

**Lemma 3**

$$
val'(k) \geq \sum_{all \ i \ | \ k \ supplies \ them \ in \ opt. \ soln.} \delta(i)
$$

**Proof.** Let us consider the portion of $val'(k)$ that comes from a single $m_i$ that $k$ serves in the optimal solution, and compare it to $\delta(i)$. This portion is at least 0. If it is 0 then $k$ is not serving
that $m_i$, so $\lambda_{i\sigma(i)} > \lambda_{i\sigma'(i)}$ in which case $\delta(i) < 0$ which is what we want. Otherwise $k$ is getting some profit from $m_i$. This profit is $p'_i - \lambda_{i\sigma'(i)}$, where $p'_i$ must be $\lambda_{i\sigma(i)}$. So we get:

$$p'_i - \lambda_{i\sigma'(i)} \geq \delta(i)$$

$$p'_i - \lambda_{i\sigma'(i)} \geq \lambda_{i\sigma(i)} - \lambda_{i\sigma'(i)}$$

$$p'_i \geq \lambda_{i\sigma(i)}$$

Thus, each individual element of $val'(k)$ is less than or equal to $\delta(i)$. So the sum over all elements must also have this property and the lemma is proven.

But this implies that $\sum_i val(k) \geq \sum_i \delta(i)$. But note that

$$\sum_{all i} \delta(i) = \sum_{all i} (\lambda_{i\sigma(i)} - \lambda_{i\sigma'(i)})$$

$$\sum_{all i} \delta(i) = \sum_{all i} (\lambda_{i\sigma(i)} - \lambda_{i\sigma'(i)} + \Pi_i - \Pi_i)$$

$$\sum_{all i} \delta(i) = \sum_{all i} (\Pi_i - \lambda_{i\sigma'(i)}) - \sum_{all i} (\Pi_i - \lambda_{i\sigma(i)})$$

$$\sum_{all i} \delta(i) = TotalBenefit(Opt.) - TotalBenefit(Nash)$$

But we know that $TotalBenefit(Nash) \geq \sum_i val(k) \geq \sum_i \delta(i)$ as the sum over $val(k)$ only considers the benefit to the suppliers. Then we have:

$$\sum_{all i} val(i) \geq TotalBenefit(Opt.) - TotalBenefit(Nash)$$

$$TotalBenefit(Nash) \geq TotalBenefit(Opt.) - TotalBenefit(Nash)$$

$$2 \ast TotalBenefit(Nash) \geq TotalBenefit(Opt.)$$

$$TotalBenefit(Nash) \geq TotalBenefit(Opt.)/2$$

\[\square\]