These simple problems are taken almost verbatim from Chapter 1 of [L04]. I added a brief discussion of representing languages by monadic second-order formulas, since we didn’t get to that in lecture on Thursday.

1 First Order Graph Properties

Show how to express the following graph properties in first-order logic:

a) A graph is complete (every vertex is connected by an edge to every other vertex).

b) A graph has at least two vertices of degree (exactly) 3.

c) Every vertex is connected by an edge to a vertex of degree (exactly) 3.

2 Existential Second Order Graph Properties

Show how to express the following graph properties in $\exists SO$ logic:

a) A graph has an independent set $X$ of size at least $k$. (A set $X$ of vertices is independent if no two vertices in $X$ are connected by an edge.)

b) A graph on $n$ vertices has an independent set of size at least $n/2$.

c) A graph has a kernel. (A set $X$ of vertices is a kernel if $X$ is independent set and every vertex not in $X$ is connected by an edge to some vertex in $X$.)

3 String Properties

Given a string $s = s_1s_2\ldots s_n$ over alphabet $\{a, b\}$, we create a structure $M_s$ as follows: The universe is $\{1, 2, \ldots, n\}$, corresponding to positions in the string. There is a single binary relation $<$ whose interpretation is the usual order on the natural numbers. There are two unary relations $A$ and $B$ such that $A(i)$ is true if $s_i = a$ and $B(i)$ is true if if $s_i = b$.

We say sentence $\phi$ defines language $L \subseteq \{a, b\}^*$ if

$$M_s \models \phi \iff s \in L$$
We may allow $\phi$ to be a first-order sentence or an existential monadic second-order sentence.

**a)** Give a definition of the regular language

$$a^*(b + c)^*aa^*$$

by a first-order sentence.

**b)** Give a definition of the regular language

$$(aaa)^*(bb)^+$$

by a monadic second-order sentence.

**c) (a bit more difficult)** Suppose you are given a finite automaton $A$ (you may assume wlog that $A$ is deterministic). Show how to construct a monadic existential second-order sentence $\phi_A$ that defines the language accepted by $A$ according to

$$M_s \models \phi_A \iff s \in L(A)$$

Note you may need quite a few second-order existential quantifiers. This is one direction of the claimed-in-lecture equivalence between monadic second-order and regular sets.