Decision Theory I
Problem Set 1

1. Show that if $\succ$ is negatively transitive and asymmetric then $\succ$ is transitive.

2. Suppose that $\succ$ is a partial, asymmetric and transitive relation on a finite set $X$. Let $c(\cdot, \succ)$ be the choice function induced by $\succ$. Does this choice function necessarily satisfy Sen’s $\beta$?

3. Suppose $X = \{x, y, z\}$. Consider a choice function $C : P(X) \rightarrow P(X)$ such that $C(\{x, y\}) = \{x\}$, $C(\{x, z\}) = \{z\}$ and $C(\{y, z\}) = \{y\}$. Does this choice function satisfy Sen’s $\alpha$ and $\beta$?

4. The set of alternatives is $X = \{a, b, c\}$ and $\succ$ is a binary order on $X$ reflecting strict preference. Suppose that for $x \in \{b, c\}$, $x \not\succ a$ and $a \not\succ x$. Suppose also that $b \succ c$. Can this relation be a strict preference relation? Explain.

   If we want to include the possibility that there is an alternative $a$ that is not comparable to either $b$ or $c$ in our analysis then we would want the condition above on $a$ to be satisfied. What does this example say about non-comparability?

5. Let $\succ$ be a binary relation on a finite set $X$. Define $\succeq$ by: $x \succeq y$ if $y \not\succ x$. Show
   
   (a) If $\succeq$ is complete then $\succ$ is asymmetric.
   
   (b) If $\succeq$ is transitive then $\succ$ is negatively transitive.

6. **GRAD**: A binary relation that is reflexive, symmetric and transitive is called an equivalence relation. An equivalence relation partitions a set into equivalence classes. Suppose that $\succ$ is a strict preference relation on a finite set $X$. Then by Proposition 2.4 of Kreps we know that $\sim$ is an equivalence relation on $X$. For each $x \in X$ define its equivalence class by $I(x) = \{y \in X| y \sim x\}$. Show:

   (a) The sets $I(x)$ partition $X$. (A collection of sets $\{A_1, \ldots, A_N\}$ partitions $X$ if each $x \in X$ is in at least one $A_i$ and $A_i \cap A_j = \emptyset$ for all $i \neq j$.)
(b) The sets $I(x)$ are strictly ranked. (The equivalence classes are strictly ranked if, for all $x, y \in X$: (1) if $I(x) \neq I(y)$, then either $x \succ y$ or $y \succ x$, and (2) if $x \succ y$ then $x' \succ y'$ for all $x' \in I(x)$ and $y' \in I(y)$.)

7. **GRAD:** In the statement of Sen’s $\alpha$ and $\beta$ we allow the sets $A$ and $B$ to be any subsets of $X$. So when we proved that these axioms imply that the revealed preference relation is asymmetric and negatively transitive we allowed ourselves to use information about choices from arbitrary subsets of $X$. We want to know whether there is a smaller class of subsets of $X$ such that the claim in the revealed preference theorem is true if $\alpha$ and $\beta$ are satisfied on this smaller class of sets. More precisely, find the smallest set $S$ of non-trivial (not just single element sets), non-empty subsets of $X$ such that the following claim is true: If a choice function satisfies Sen’s $\alpha$ and $\beta$ on $S$ then there is a preference order $\succ$ defined on $X$ such that $c(A, \succ) = c(A)$ for all $A \in S$.

8. **GRAD:** In class in the proof of the revealed preference theorem we defined strict revealed preference. Weak revealed preference is defined as follows: $x \succeq y$ if $x \in C(\{x, y\})$. Define induced strict revealed preference $\succ^*$ from revealed preference $\succeq$ by: $x \succ^* y$ if $x \succeq y$ and $y \not\succeq x$. Are strict revealed preference and induced strict revealed preference the same relation?